

Tax Impact on Multistage Mean-Variance Portfolio Allocation

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Abstract

We investigate the sensitivity to tax change of multistage portfolio allocation over a discrete time investment horizon. Special taxation rules within wrappers grouped a number of risky assets are integrated with multistage linear or quadratic stochastic programming in the mean-variance framework. The uncertainty on the returns of assets is specified as a scenario tree generated by simulation based approach. We adjust different values on capital gains tax under different asset bounds and risk levels. The tax impact in the yearly reallocation of the investments for a typical case with an annual fixed withdrawal that utilizes completely the option of taper relief is also explored. Our computational results show that taxes, combined with other effects such as risk and investment upper bounds, have a significant performance impact on portfolio allocation as well as diversification over wrappers. Yet, investment strategies can be made robust to changes in taxation.

Keywords Finance, stochastic programming, quadratic programming, risk management, scenarios

1 Introduction

In this paper, we investigate the impact of changes in taxation on optimal investment strategies. We perform sensitivity analysis for changes in capital gains tax, upper bounds on investment assets and risk levels. It is well known that taxation of returns on financial assets alters the benefits of saving for future consumption and thus affects the trade-off between current consumption and investment. Moreover, the ability of investors to defer the taxation of capital gains or tax on wrappers at the end of the horizon alters the relative valuation of stocks and bonds and, thus affects the optimal portfolio composition of investors.

This research utilizes a multistage mean-variance optimization model that maximizes total expected wealth and minimizes expected risk at the end of investment horizon [6, 26].

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Multistage linear or quadratic stochastic programming is used to model the portfolio allocation problem; see for example [8, 15, 16]. Uncertainty on future behaviour of the asset returns is represented by a scenario tree. A scenario tree is a discrete approximation of a multivariate continuous distribution [11, 12, 14]. In multistage linear stochastic programming models, the approximate nature of the discrete set of scenarios can be improved using a risk representation. The risk is represented by the variance of portfolio returns [16].

Taxes play an important role in the decisions of individual investors. This is often larger than transaction costs or any other effect for many investors. It is important for an investor, to consider the effect of taxes on all investment decisions, including optimal asset allocation. Post-tax optimization determines a discrete-time optimal after-tax asset allocation over an investment horizon within the mean-variance framework. The specific tax rules within a wrapper (defined as a collection of assets) are integrated with multistage mean-variance optimization to yield an overall post tax risk return efficient investment strategies.

In [19], we develop a post-tax model for investments. In [20], we consider stochastic linear versus stochastic mixed integer linear programming models for post-tax portfolio optimisation. The present study considers a different approach which integrates risk, measured in terms of post-tax portfolio variance, and provides a tool for evaluating the best strategy in view of the risk aversion of the decision maker. It also provides an extensive sensitivity study of tax changes versus attitudes to risk. In [?], we consider detailed linear and quadratic (mean-variance) mixed-integer programming formulations for the post-tax model incorporating the general ability to withdraw from capital as required. In the present paper, we restrict withdrawals each year to the amount of post-tax income during the year and provide a detailed study of tax-impact.

Feldstein and Hubbard considered the relation between taxation and portfolio composition in [7], and [13]. The trade-off between seeking maximum expected return and minimum risk while taking into account the tax consequences of investment decisions does not seem to have received sufficient attention. There are well established methods for pre-tax portfolio management. However, for portfolio management in view of taxation, or portfolio performance evaluation after-tax, does not seem to have been studied.

Stein has proposed a technique to measure a portfolio value after tax in [22]. Furthermore, in [23] Stein has investigated the diversification dilemma in the presence of taxes and the sensitivity of key parameters such as the initial holding and investment horizon, without an integrated approach to taxation and the determination of optimal portfolios. The relation between risk and capital income tax has been investigated by Asea et. al [1] and the benefits of risk diversification in multiple stock portfolios by Vassal [24]. Dammon et. al characterized the optimal dynamic consumption and portfolio decisions in the presence of capital gains taxes and short-sale restrictions, [4] and [5]. They evaluated the optimal decisions as a function of the investor's age, initial portfolio holdings, and tax basis. Only fixed return values and direct assets investment are considered in these. The present study is the first of its type to introduce wrappers and forecast returns scenarios with associated risk. Our computational results show that taxes, combined with other effects such as risk and investment upper bounds, have a significant performance impact on portfolio allocation, diversification over wrappers as well as asset distribution. The paper is organized as follows. In Section 2, we present the notation and stochastic linear and quadratic programming models. Section 3 explains our computational experiments. Results are reported in Section 4.

2 Multistage Mean-Variance Model

2.1 Notation

The notation used for post-tax portfolio optimization is described in Table 1. We represent vectors in \mathbb{R}^n in boldface. The transpose of a vector and matrix is denoted by the symbol $'$. The subscript indicates either time period, or events.

2.2 Scenario tree

We consider n risky assets and construct a portfolio over an investment horizon T . The portfolio is restructured over a period in terms of both return and risk. After the initial investment ($t = 0$), the portfolio may be restructured at discrete times $t = 1, \dots, T - 1$, and redeemed at the end of the period ($t = T$).

The random variables are the uncertain return values of each asset on an investment. The discretization of the random values and the probability space leads to a framework in which a random variable takes finitely many values. Thus, the factors driving the risky events are approximated by a discrete set of scenarios, or sequence of events. Given the event history up to a particular time, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observation. This branching process is represented using a scenario tree. See [20] for more specific details of the description of the scenario tree, but a summary as follows. A scenario is a possible realisation of the stochastic variables. Hence, the set of scenarios corresponds to the set of leaves of the scenario tree, \mathcal{N}_T , and nodes of the tree at level $t \geq 1$ (the set \mathcal{N}_t) correspond to possible realisations of $\boldsymbol{\rho}^t$. The set of all interior nodes of the scenario tree is \mathcal{N}_I . A node of the tree is denoted by $\mathbf{e} = (s, t)$, where s is a scenario (path from root to leaf), and time period t specifies a particular node on that path.

At each node of time period t , decisions for weights of each asset, transactions for buying and selling \mathbf{w}_t , \mathbf{b}_t , \mathbf{s}_t , respectively, must be determined. Due to the recourse nature of the multistage problem, decision variables \mathbf{w}_t , \mathbf{b}_t and \mathbf{s}_t are influenced by previous stochastic events $\boldsymbol{\rho}^t$, and hence $\mathbf{w}_t = \mathbf{w}_t(\boldsymbol{\rho}^t)$, $\mathbf{b}_t = \mathbf{b}_t(\boldsymbol{\rho}^t)$, and $\mathbf{s}_t = \mathbf{s}_t(\boldsymbol{\rho}^t)$. For simplicity, we shall use the terms \mathbf{w}_t , \mathbf{b}_t and \mathbf{s}_t , and assume their implicit dependence on $\boldsymbol{\rho}^t$. Notice that $\boldsymbol{\rho}_t$ can take only finitely many values.

In this paper, we only consider the simulation based approach to generate the scenario tree. For further details, the reader is referred to [14].

2.3 Tax Rules and Wrappers

A wrapper is a set of assets with a specific set of tax rules on a regular basis and in different investor scenarios. We consider the UK tax rules within three wrappers, namely offshore bond, onshore bond and unit trust. Investments in each wrapper may include equities, bond, cash and property. The superscript for the type of wrappers $k = 1, 2, 3$ for offshore bond, onshore bond and unit trust, respectively is used in the problem formulation. The description and rules regarding the tax properties of the wrappers are specified by Chadwick-Healey et. al in [3].

Symbols and Input Data	
$\mathbf{1}$	$\equiv (1, 1, 1, \dots, 1)'$
$\mathbf{u}'\mathbf{v}$	$\equiv u_1v_1 + u_2v_2 + \dots + u_nv_n$ (Inner product)
$\mathbf{u} \circ \mathbf{v}$	$\equiv (u_1v_1, u_2v_2, \dots, u_nv_n)'$ (Hadamard product)
n	number of investment assets.
T	investment planning horizon
\mathbf{L}	Lower percentage bounds
\mathbf{U}	Upper percentage bounds
$\boldsymbol{\rho}_t$	vector of stochastic data observed at time t , $t = 0, \dots, T$
$\boldsymbol{\rho}^t$	history of stochastic data up to t
\mathcal{N}_t	set of nodes of the scenario tree at time t
s	index denoting a scenario (path from root to leaf)
$\mathbf{e} \equiv (s, t)$	index denoting an event node of the scenario tree
$a(\mathbf{e})$	ancestor of event $\mathbf{e} \in \mathcal{N}$ (parent in the scenario tree)
$p_{\mathbf{e}}$	branching probability of event \mathbf{e} : $p_{\mathbf{e}} = \text{Prob}[\mathbf{e} a(\mathbf{e})]$
$P_{\mathbf{e}}$	probability of event \mathbf{e} : if $\mathbf{e} = (s, t)$, then $P_{\mathbf{e}} = \prod_{i=1\dots t} p_{(s,i)}$
$E[\cdot]$	Expectation with respect to ρ
$\Lambda_c \in \mathbb{R}^{n \times n}$	covariance matrix of capital gains
$\Lambda_d \in \mathbb{R}^{n \times n}$	covariance matrix of dividends or returns
$\mathbf{c}_t(\boldsymbol{\rho}^t)$	percentage capital gains, $t = 1, \dots, T$
$\mathbf{d}_t(\boldsymbol{\rho}^t)$	percentage dividends or returns $t = 1, \dots, T$.
$\hat{\mathbf{c}}_{\mathbf{e}}$	expectation of $\mathbf{c}_t(\boldsymbol{\rho}^t)$ for event \mathbf{e} , conditional on $\boldsymbol{\rho}^{t-1}$
$\hat{\mathbf{d}}_{\mathbf{e}}$	expectation of $\mathbf{d}_t(\boldsymbol{\rho}^t)$ for event \mathbf{e} , conditional on $\boldsymbol{\rho}^{t-1}$
$\mathbf{c}_{\mathbf{e}}$	realization of \mathbf{c}_t in event \mathbf{e} : $\mathbf{c}_{\mathbf{e}} \sim N(\hat{\mathbf{c}}_t(\boldsymbol{\rho}^t), \Lambda_c)$
$\mathbf{d}_{\mathbf{e}}$	realization of \mathbf{d}_t in event \mathbf{e} : $\mathbf{d}_{\mathbf{e}} \sim N(\hat{\mathbf{d}}_t(\boldsymbol{\rho}^t), \Lambda_d)$
Tax Rates	
t_{gb}	tax on gross returns of offshore bond
t_{ob}	tax within onshore bond
t_{gr}	tax at the end on gross returns from onshore bond
\mathbf{t}_{inc}	vector of income tax paid on dividends or income
CG_t	capital gains tax at period t
ic_k	initial cost for wrapper k (as percentage of investment in wrapper)
ac_k	annual cost for wrapper k (as percentage of investment in wrapper)
Decision Variables	
W_*	total portfolio value, across wrappers
\mathbf{w}_*^+	amount of money held in each asset
\mathbf{b}_*^+	amount bought of each asset
\mathbf{s}_*^+	amount sold of each asset
\mathbf{h}_*^+	tax-deferred or tax-free withdrawals
\mathbf{g}_*^+	withdrawals subject to immediate tax
R_*^+	cumulative returns

Table 1. Notation

Returns are obtained as dividends, income, and capital gains from instruments within each wrapper. Dividend is the percentage return obtained from equities and bonds and income for cash assets or properties. The capital gains is the percentage growth in the capital value of the assets in the portfolio.

Taxes are payable in different ways for different wrappers and assets. Taxes are imposed in specific situations according to the wrapper utilized. The main taxes applicable in this paper are income, capital gains, tax within an onshore bond wrapper and tax on gross returns. In addition, we assume that an investor is allowed to withdraw some amount of money from different wrappers with different tax regulations.

Income tax, t_{inc} , is paid annually in unit trust wrapper for the returns received at that year as 25% for UK dividend income 32.5% for non UK dividend income and 40% for the interest income.

Profit obtained by disposal of certain types of assets and wrappers, then the capital gains tax is subject to capital gains tax. Taper relief, a gradual reduction of the amount of tax payable, is also provided depending on the length of time shares or property has been held. The longer the time horizon until encashment, the lower tax is payable. If it is for three years or less, then 40% capital gains tax is payable. From fourth year onwards, tax reduces by 2% per annum. After eleven years, the maximum taper relief is achieved as 24%.

Tax charge in an onshore bond wrapper arises annually withing a fund, which is usually 22%. Tax on gross returns is paid on the encashment of onshore and offshore bond wrappers. Tax of 18% is due for any income and gains arising are in onshore bond. In offshore bond there is no tax arise on income or gains until on encashment; but the tax rate is of 40% on encashment.

Withdrawals may only be taken from the gains since the original capital must remain in the wrapper until encashment. In onshore and offshore bond, investors can withdraw 5% of initial capital with tax defer until the end of investment horizon. Unused portions of the annual allowance of 5% may be carried forward, i.e. if no withdrawals are taken until year four, then tax deferred withdrawals as large as 20% of original investment is allowed. Withdrawals beyond the cumulative 5% limit may also be taken, but are subject to immediate tax at the encashment rate of t_{gb} .

In unit trust wrapper, withdrawals may be taken from the current year's growth only. As dividend and income returns have their taxes paid annually, withdrawals can be taken from them without further tax. However, withdrawals from growth due to capital gains are subject to tax.

The bank claims an annual percentage fee (ac_k) from the total value of any wrapper and charge a percentage for initial setup (ic_k) for a wrapper $k = 1, 2, 3$.

2.4 Stochastic Linear Programming (SLP) Model

Many traditional multistage portfolio analysis programs seek only to maximize expected return. Referencing previous results, this can be achieved with a linear programming formulation; for more information on stochastic programming, the reader is referred to [2, 9, 10, 21, 26]. For post-tax optimization the strategy implemented is the classical stochastic programming approach (SLP) which incorporates the mean term. This is a risk-neutral approach which does not take risk-attitudes into account.

The SLP Model

$$\text{maximize Mean} = \sum_{k=1}^3 E[NRV_k]$$

subject to

$$W_0 = \sum_{k=1}^3 \mathbf{1}' \mathbf{w}_0^k$$

$$\mathbf{1}' \mathbf{b}_e^k - \mathbf{1}' \mathbf{s}_e^k = 0, \quad \mathbf{e} \in \mathcal{N}_t, \quad t = 1, \dots, T-1, \quad k = 1, 2, 3$$

$$E[NRV_1] = \sum_{\substack{\mathbf{e} \in \mathcal{N}_T \\ t=0}}^{T-1} \left[P_e(1 - ac_1)(\mathbf{1} + \hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^1 \right] \\ - \sum_{t=0}^{T-1} \sum_{\mathbf{e} \in \mathcal{N}_t} \left[-P_e t_{gb} \left((1 - ac_1)(\hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^1 - \frac{1}{1 - t_{gb}} \mathbf{1}' \mathbf{g}_e^1 \right) \right]$$

$$E[NRV_2] = \sum_{\substack{\mathbf{e} \in \mathcal{N}_T \\ t=0}}^{T-1} \left[P_e(1 - ac_2)(\mathbf{1} + \hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^2 \right] \\ - \sum_{t=0}^{T-1} \sum_{\mathbf{e} \in \mathcal{N}_t} \left[-P_e t_{gr} \left((1 - ac_2)(\hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^2 - \frac{1}{1 - t_{gr}} \mathbf{1}' \mathbf{g}_e^2 \right) \right]$$

$$E[NRV_3] = \sum_{\substack{\mathbf{e} \in \mathcal{N}_T \\ t=0}}^{T-1} \left[(1 - ac_3) (\mathbf{1} + \hat{\mathbf{c}}_e + (1 - t_{inc}) \circ \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^3 \right] \\ - \sum_{t=0}^{T-1} \sum_{\mathbf{e} \in \mathcal{N}_t} \left[-P_e CG_T \left((1 - ac_3) \hat{\mathbf{c}}_e' \mathbf{w}_{a(e)}^3 - \frac{1}{1 - CG_e} \mathbf{1}' \mathbf{g}_e^3 \right) \right]$$

For $\mathbf{e} \in \mathcal{N}_t, t = 1, \dots, T-1$

$$\mathbf{w}_e^1 = (1 - ac_1) (\mathbf{1} + \hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e) \circ \mathbf{w}_{a(e)}^1 + (1 - tc) \mathbf{b}_e^1 - (1 + tc) \mathbf{s}_e^1 - \mathbf{h}_e^1 - \frac{1}{1 - t_{gb}} \mathbf{g}_e^1$$

$$\mathbf{w}_e^2 = (1 - ac_2) (\mathbf{1} + (1 - t_{ob})(\hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)) \circ \mathbf{w}_{a(e)}^2 + (1 - tc) \mathbf{b}_e^2 - (1 + tc) \mathbf{s}_e^2 - \mathbf{h}_e^2 - \frac{1}{1 - t_{gr}} \mathbf{g}_e^2$$

$$\mathbf{w}_e^3 = (1 - ac_3) (\mathbf{1} + \hat{\mathbf{c}}_e + (1 - t_{inc}) \circ \hat{\mathbf{d}}_e) \circ \mathbf{w}_{a(e)}^3 + (1 - tc) \mathbf{b}_e^3 - (1 + tc) \mathbf{s}_e^3 - \mathbf{h}_e^3 - \frac{1}{1 - CG_e} \mathbf{g}_e^3$$

$$R_e^1 = R_{a(e)}^1 + (1 - ac_1) (\hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^1 - \mathbf{1}' \mathbf{h}_e^1 - \frac{1}{1 - t_{gb}} \mathbf{1}' \mathbf{g}_e^1$$

$$R_e^2 = R_{a(e)}^2 + (1 - ac_2)(1 - t_{ob}) (\hat{\mathbf{c}}_e + \hat{\mathbf{d}}_e)' \mathbf{w}_{a(e)}^2 - \mathbf{1}' \mathbf{h}_e^2 - \frac{1}{1 - t_{gr}} \mathbf{1}' \mathbf{g}_e^2$$

$$\sum_{e'=0}^e \mathbf{1}' \mathbf{h}_e^k \leq (0.05t) \mathbf{1}' \mathbf{w}_0^k \quad \mathbf{e} \in \mathcal{N}_t, t = 1, \dots, T-1 \quad k = 1, 2$$

$$\mathbf{h}_e^3 \leq (1 - ac_3)(\mathbf{1} - t_{inc}) \circ (\hat{\mathbf{d}}_e \circ \mathbf{w}_{a(e)}^3) \quad \mathbf{e} \in \mathcal{N}_t, t = 1, \dots, T-1$$

$$\mathbf{g}_e^3 \leq (1 - ac_3)(1 - CG_e)(\hat{\mathbf{c}}_e \circ \mathbf{w}_{a(e)}^3) \quad \mathbf{e} \in \mathcal{N}_t, t = 1, \dots, T-1$$

$$TW_t = \sum_{k=1}^3 \mathbf{1}' \mathbf{h}_e^k + \mathbf{1}' \mathbf{g}_e^k, \quad \mathbf{e} \in \mathcal{N}_t, t = 1, \dots, T-1$$

$$W_e = \sum_{k=1}^3 \mathbf{1}' \mathbf{w}_e^k \quad \mathbf{e} \in \mathcal{N}_t, t = 0, \dots, T-1$$

$$\sum_{k=1}^3 \mathbf{w}_e^k \geq \mathbf{L} W_e \quad \mathbf{e} \in \mathcal{N}_t, t = 0, \dots, T-1$$

$$\sum_{k=1}^3 \mathbf{w}_e^k \leq \mathbf{U} W_e \quad \mathbf{e} \in \mathcal{N}_t, t = 0, \dots, T-1$$

$$\mathbf{w}_e^k, \mathbf{b}_e^k, \mathbf{s}_e^k, \mathbf{g}_e^k, \mathbf{h}_e^k, R_e^k \geq 0 \quad \mathbf{e} \in \mathcal{N}_t, \quad t = 0, \dots, T, \quad k = 1, 2, 3$$

$$W_e \geq 0 \quad \mathbf{e} \in \mathcal{N}_t, \quad t = 1, \dots, T$$

The linear constraints in the SLP model describe the allocation of initial investment, the computation of expected net redemption for each wrapper, money growth and profit transfers between assets at each time period, accumulation of returns, and withdrawals as well as the bounds on decision variables.

Notice that the annual bank fees deducted by term $(1 - ac_k)$ for $k = 1, 2, 3$ must be augmented by the bank's initial setup fees in the first year. For $\mathbf{e} \in \mathcal{N}_1$ (children of the root scenario node, the terms becomes $(1 - ic_k - ac_k)$.

The objective function is to maximize total expected net redemption value obtained from each wrapper, which can be balanced to find efficient post-tax investment allocations. For more details of the SLP model, the reader is referred to [20].

2.5 Stochastic Quadratic Programming (SQP) Model

The SQP approach attempts to inject risk aversion into the optimization model. It incorporates the quadratic variance term and permits the minimization of the variability of the terminal wealth given observed statistics. This assumes that the investor is risk averse and ensures a degree of risk aversion by the investor by relaxing the certainty of the return values along a given leaf of the scenario tree. The other objectives that naturally lead to diversification by taking risk-attitudes into account such as utility and minmax have been described in [27, 25].

An important assumption underlying the mean variance approach adopted is the the normality of the return distributions. This is clearly not the case for asset classes such as property and bonds. However, for investments of long term maturity and for indices of the corresponding asset class, we assume this to be an acceptable approximation as far as risk assessment is concerned.

The variance of wealth at time t of a particular wrapper k can be calculated as

$$\text{Var} [\mathbf{1}' \mathbf{w}_t^k] = \text{Var} [(c^k \mathbf{c}_t + d^k \mathbf{d}_t)' \mathbf{w}_{t-1}^k] \quad (1)$$

$$= \text{E} \left[((c^k \mathbf{c}_t + d^k \mathbf{d}_t)' \mathbf{w}_{t-1}^k)^2 \right] - \left(\text{E} [(c^k \mathbf{c}_t + d^k \mathbf{d}_t)' \mathbf{w}_{t-1}^k] \right)^2 \quad (2)$$

$$\equiv \text{E} \left[\left((c^k \mathbf{c}_t + d^k \mathbf{d}_t)' \mathbf{w}_{t-1}^k \right)^2 \right]$$

$$= \text{E} \left[\mathbf{w}_{t-1}^{k'} \left(c^k \mathbf{c}_t + d^k \mathbf{d}_t \right) \left(c^k \mathbf{c}_t + d^k \mathbf{d}_t \right)' \mathbf{w}_{t-1}^k \right]$$

$$= \text{E} \left[\mathbf{w}_{t-1}^{k'} \left((c^k)^2 \mathbf{c}_t \mathbf{c}_t' + 2c^k d^k \mathbf{c}_t \mathbf{d}_t' + (d^k)^2 \mathbf{d}_t \mathbf{d}_t' \right)' \mathbf{w}_{t-1}^k \right]$$

$$= \text{E} \left[\mathbf{w}_{t-1}^{k'} \left((c^k)^2 \Lambda_c + (d^k)^2 \Lambda_d \right) \mathbf{w}_{t-1}^k \right] \quad (3)$$

$$= \sum_{\mathbf{e} \in \mathcal{N}_t} P_{\mathbf{e}} \left[\mathbf{w}_{a(\mathbf{e})}^{k'} \left((c^k)^2 \Lambda_c + (d^k)^2 \Lambda_d \right) \mathbf{w}_{a(\mathbf{e})}^k \right] \quad (4)$$

where c^k and d^k are scalar factors for capital gains \mathbf{c}_t and dividends or income returns \mathbf{d}_t , respectively. The total return is expressed in terms of capital gains and dividends or income returns. These scalar constants are determined by the nature of annual taxation in the wrapper. The following table lists the values for all wrappers

Notice that for any random vector \mathbf{y} , $\text{E}[\mathbf{y}\mathbf{y}']$ is equivalent to the covariance matrix of \mathbf{y} — as well as the assumption that capital gains (\mathbf{c}) are independent from dividends or income returns (\mathbf{d}), which eliminates the term $\text{E}[\mathbf{c}_t \mathbf{d}_t'] = 0$.

Wrappers		c^k	d^k
offshore bonds	k=1	$(1 - ac_1)$	$(1 - ac_1)$
onshore bonds	k=2	$(1 - ac_2)$	$(1 - ac_2)$
unit trust	k=3	$(1 - ac_3)$	$(1 + t_{inc})(1 - ac_3)$

Financial reality dictates that the highest-performing portfolio strategy is also the most risky efficient strategy available. In order to obtain other points on the Markowitz efficient frontier, it is necessary to consider risk (variance), in conjunction with the mean return. In this case, the required expected net redemption value can be provided as constant \mathcal{W}_T . The SQP problem whose optimum is the efficient (least risky) multistage investment strategy can be outlined as the following optimization problem.

The SQP Model

$$\text{minimize Variance} = \sum_{t=0}^{T-1} \sum_{k=1}^3 \sum_{\mathbf{e} \in \mathcal{N}_t} P_{\mathbf{e}} \left[\mathbf{w}_{a(\mathbf{e})}^{k'} \left((c^k)^2 \Lambda_c + (d^k)^2 \Lambda_d \right) \mathbf{w}_{a(\mathbf{e})}^k \right]$$

subject to

Constraints of SLP problem

$Mean \geq \mathcal{W}_T$

The optimal investment strategy yields expected post-tax wealth of \mathcal{W}_T subject to other linear constraints defined in SLP problem. By varying \mathcal{W}_T from a risk-seeking strategy (obtainable by the SLP presented in Section 2.3) to a risk-averse strategy (obtainable by optimizing the above SQP without the performance constraint, $Mean \geq \mathcal{W}_T$), the efficient frontier can be generated. In general terms, the efficient frontier is obtained as follows. The maximum-mean SLP problem is first solved to find the risk-seeking strategy; that is also the maximum expected net redemption value, \mathcal{W}_{\max} . The minimum expected net redemption value, \mathcal{W}_{\min} , is then computed by solving the SQP problem without the performance constraint, $Mean \geq \mathcal{W}_T$. Its optimal constructs the risk-averse strategy. For a number of equally-spaced points $\mathcal{W}_T \in [\mathcal{W}_{\min}, \mathcal{W}_{\max}]$, the corresponding $SQP(\mathcal{W}_T)$ is optimized. The different investment strategies at different risk levels generates the points along the efficient frontier.

3 Computational Experiments

3.1 Implementation

The post-tax mean-variance optimization model explained in the previous section is implemented and integrated with a software package called *posttax*. *Posttax* is written in C++ and uses the interior point linear/quadratic solver BPMPD [17] to optimize the linear and quadratic programming problems. *Posttax* has the ability to handle simple box constraints on the decision variables, as well as percentage constraints. The scenario trees input to the program can have arbitrary branching structure and depth (limited only by computer memory). All computational experiments are carried out on a 500 MHz Pentium III, running Linux with 256Meg of RAM.

3.2 Case Study

In order to illustrate tax impact on the performance of the post-tax mean-variance optimization model, we consider a classic example in which an investor who has just sold a major business and wishes to invest efficiently for the future, perhaps towards retirement. Annual withdrawals are imposed, along with a choice of investment horizon that activates taper relief for the capital gains tax. The optimal investment strategies are computed using a scenario tree generated by simulation [14].

An investor has sold a factory and would like to invest £10,000,000 in a bank for the next eleven years. He would like to get an annual withdrawal of £500,000 for the next 10 years. How would he have to invest his wealth in order to maximize the amount he will receive at the end of the eleven years period?

3.3 Input Data

For the data used in the case study, the statistical parameters were measured from historical data: 151 monthly valuations of cash, bonds, and equities in the UK market, from 1988 to mid-2000. The historical data of the three assets (cash, bonds, and equities) was fit to an exponential growth curve. The obtained monthly growth rates were annualized, and used to simulate future growth. A covariance matrix was measured from the residuals of the exponential fit, and used in the simulation based scenario generation method (after assuming cash is risk-free).

A scenario tree with different values in income or dividend and capital gains is an input to the post-tax mean-variance optimization models. We generate a scenario tree with four branching at the first year and then only one branching over 10 years using simulation based approach. The event tree with this topology has 44 nodes and 4 final scenarios. The four-scenario tree is one year a head forecast for the whole periods.

The investment horizon has been considered as 11 years since beyond year 11, the capital gains tax (paid for the capital gains in unit trust) remains unchanged and no more taper relief can be achieved. The length of the investment is a very important fact for the distribution to be chosen. Time periods shorter than 11 years do not make use of the complete taper relief available in unit trust and very longer periods with large withdrawals will make full use of the facilities offered by offshore or onshore bonds about tax deferred withdrawals at the beginning of the investment plan. Besides, the model is highly sensitive to any small change in taxes, costs and bond values, and amount of withdrawals. Input data for taxes used for this example is summarized in Table 2. For scenario trees obtained with simulation and clustering, return values presented in Table 3 are used as seed for returns in the trees and obtained by fitting the 11 years of historical data to exponential growth curve. In addition, we assume 1.15% annual and 1% transaction costs and zero initial cost.

TAXES		Amount (%)
offshore bond	end of horizon	40
onshore bond	annually	22
onshore bond	end of horizon	18
income tax	cash	40
income tax	equities, bond	25
capital gain	every year	40, 40, 40, 38, 36, 34, 32, 30, 28, 26, 24

Table 2. The input data for taxes

RETURNS		
	Income or Dividend	Capital Gains
Equities	3.47	10.40
Bonds	2.72	8.16
Cash	8.34	0.00

Table 3. The input data for returns

An upper bound of 43% ($33\% = 1/3$ for equal proportion plus 10% for more flexibility) for the total amount of money invested in each asset (independently of the wrapper) is considered for the computational calculations.

Different percentages in risk level are used to reflect investor’s different attitudes to risk.

4 Computational Results

In our computational experiments, we consider a number factors which have an impact on the post-tax optimization model presented in Section 2. These factors can be summarized as follows;

- different values of capital gains tax
- different risk levels
- upper bounds

In order to investigate the tax impact on wrapper and asset allocation, we set all tax parameters in our post-tax optimization model as zero. This is the “tax-free” case. The “full taxation” case takes all tax rules into account as well as taper relief for capital gains taxes. We explore the impact of imposed bounds on asset weights, different rates of capital gain tax and risk levels.

Tax-free case

We present wrapper distribution and asset allocation for each wrapper with bounds at 43% and risk levels at the lowest (selected as 25%) and the highest (specified as 100%) in Figures 1 and 2, respectively. The results in Figure 1 and Figure 2 indicate that, while the model gives some diversification over wrappers, asset allocation for each wrapper has diversified and the structure of diversification on assets remains the same for all wrappers.

In order to show the impact of bounds on post-tax optimization models with different risk levels, we set investment upper bounds on assets as 1. In practice this is equivalent to not imposing any bounds except the budget-limit. Wrapper allocation and the asset allocation within offshore bonds (arbitrarily selected as an example) at different risk levels (such as 25%, 50%, 75%, 100% at each row, respectively) are presented in Figure 3.

We find that optimal investment strategies are robust to changes in taxation over wrappers for the risk-averse investor (at low risk such as 25% and 50%). That is the investment

strategy does not change appreciably when the tax levels change. However, if the risk level is increased towards more risky investments such as 75% or 100%, we observe some sensitivity.

Full Taxation case

We consider the influence of full taxation and bounds at different levels of risk. The results of wrapper diversification and asset allocation within Offshore bonds with and **without** bounds at different risk levels such as 25%, 50%, 75%, 100% are presented in Figures 4 and 5, respectively. As expected, bounds play a decisive role in diversification over wrappers and assets for any level of risk. In case of no upper bounds, it is the risk aversion factor that mostly influences investment decisions. This is independent of tax values.

Comparing the results in Figure 3 and Figure 5 (with a tax-free and a full taxation scenarios respectively), we observe that wrapper allocation is always diversified. However, for asset allocation, there is a turning point according to the risk level in the investor's decision. The risk-averse individual will invest all money in cash while the risk-seeker will choose equities. A more balanced attitude at 50% or 75% of risk level will tend to diversify over assets.

Capital Gains Tax (CGT)

We also investigate the sensitivity to changes in the values of capital gains tax on post-tax optimization models. We consider the minimum (24%) and maximum (40%) as well as the average capital gain tax rate (33.45%), as fixed rates over the investment horizon. The results in Figures 6 and 7 are obtained by considering the lowest 25% (the left side) and highest 100% risk levels (the right side) with minimum (the first row), average (the second row) and maximum (the last row) capital gains tax rate. We have done a sensitivity analysis in terms of diversification over wrappers in Figure 6 and asset allocations within Offshore Bond (arbitrarily selected) in Figure 7.

The results indicate that the overall allocation for wrappers and assets is robust to changes in the value of capital gains tax for risk averse investors (at 25% risk level). However, there is great sensitivity to those changes for the risk-seeking investor (100% risk level).

5 Conclusions

Taxation generally affects the net redemption value of investment strategies. However, under certain circumstances, the strategy (i.e. the proportion of current total wealth invested in a given asset class), itself might be robust to changes in taxation. Such robustness will be the result of imposing upper bounds on the proportion of the wealth invested in a given asset in order to impose diversification or the specification of a high degree of risk aversion (low risk investment policies) in the multi-period Markowitz framework of this paper. The latter two entail diversification. The best strategy is determined in view of the withdrawals and other requirements of the investor, and taper relief related to capital gains tax. High levels of risk aversion seem to render the optimal strategy insensitive (i.e. robust) to changes in taxation. This is assisted by the imposition of investment upper bounds.

Without any explicit upper bounds imposed on investments, an increased level of risk aversion seems to have a dampening effect on the response of the investment strategy to changes in taxation. For example, at a high degree of risk aversion, we have observed the ordinary tendency for investing in cash, which does not attract capital gains tax, and therefore an increasing in the latter tax would not affect the strategy or distribution over wrappers. This robustness disappears on the more risk seeking end. Investment in more risky asset classes such as equities and bonds attracts capital gains. This activates greater use of the offshore bond wrapper. An increase of capital gains tax leads to a migration from offshore bond to unit trust.

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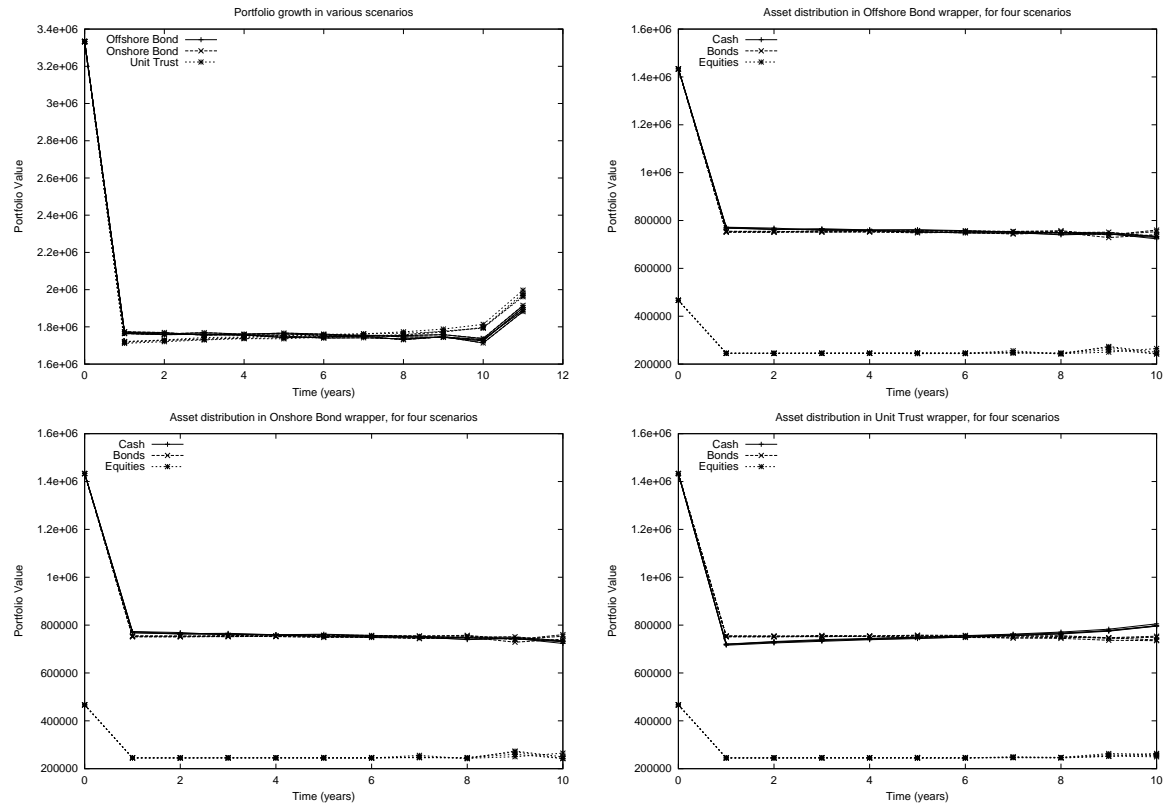


Figure 1: Tax-free case with bounds: wrapper and asset allocation at 25% risk level.

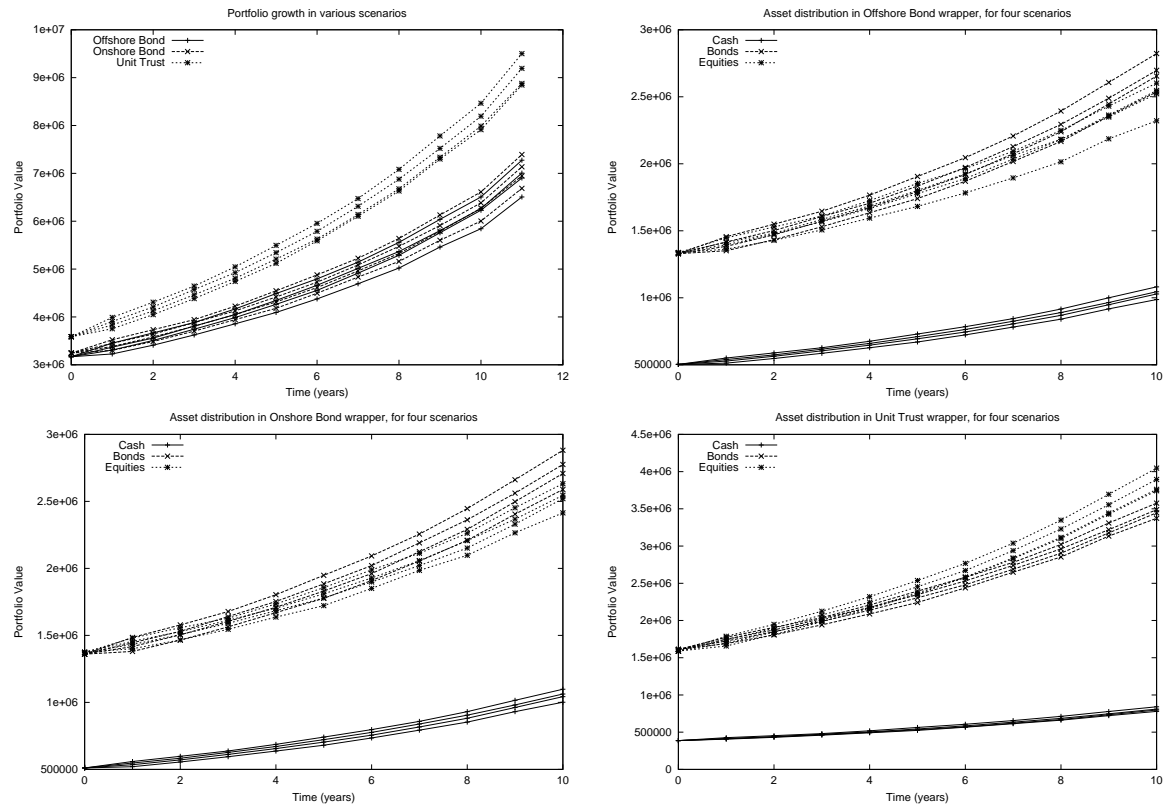
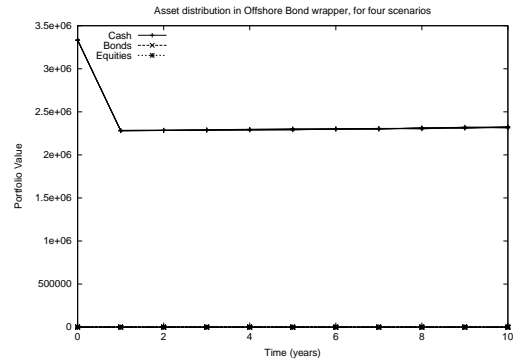
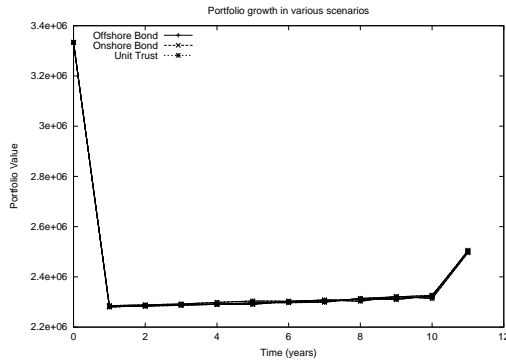
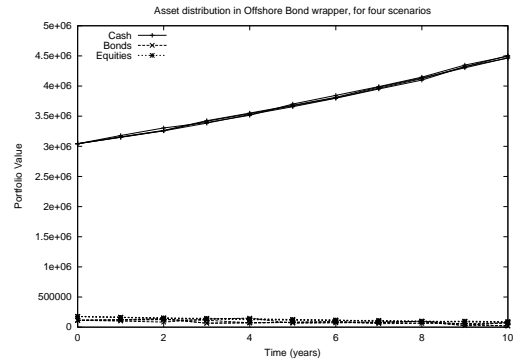
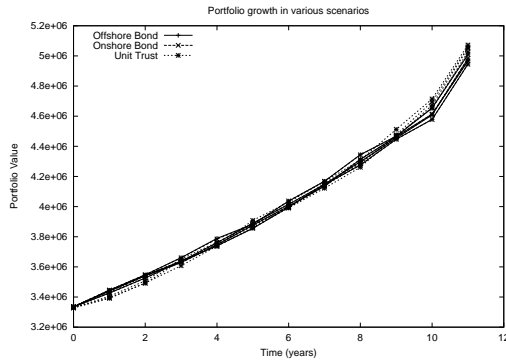


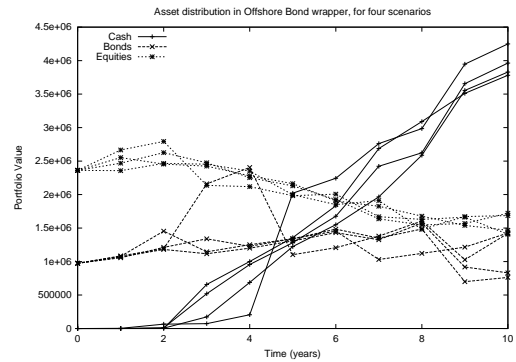
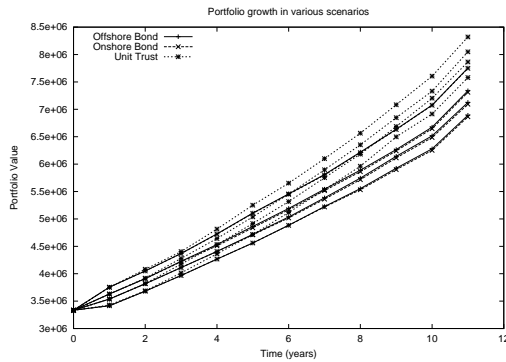
Figure 2: Tax-free case with bounds: wrapper and asset allocation at 100% risk aversion level. **Portfolio growth in various scenarios:** Total portfolio growth for three wrappers for the four scenarios



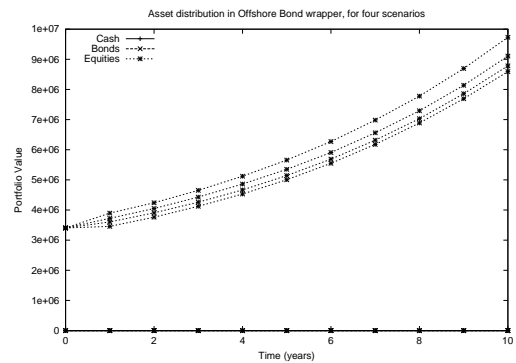
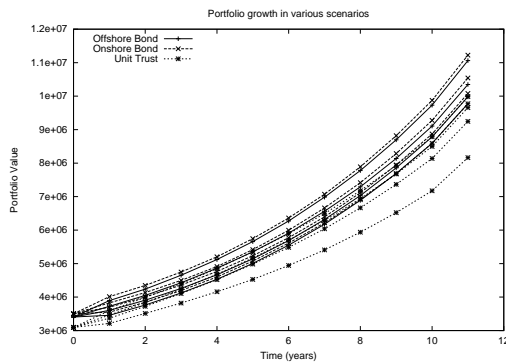
25% risk level



50% risk level



75% risk level



100% risk level

Figure 3: Tax-free case with no bounds: wrapper diversification and asset allocation for offshore bond. **Portfolio growth in various scenarios:** Total portfolio growth in three wrappers for the four scenarios

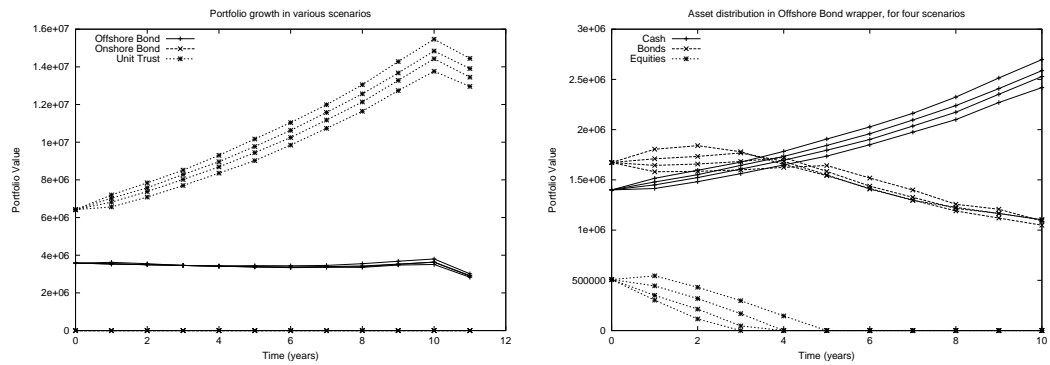
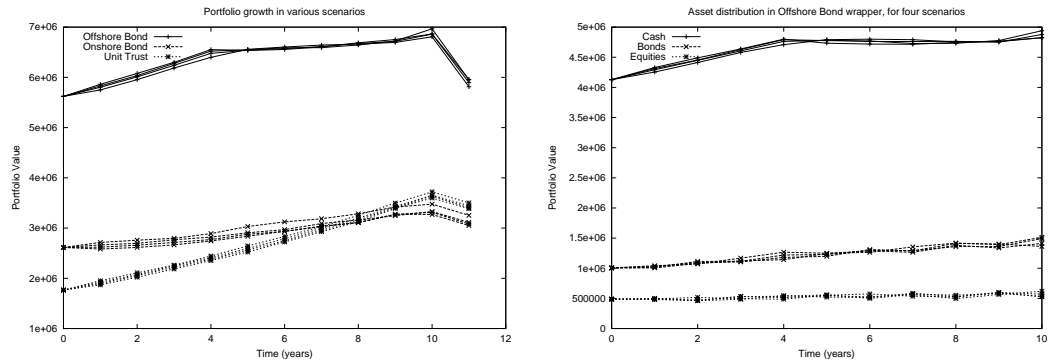
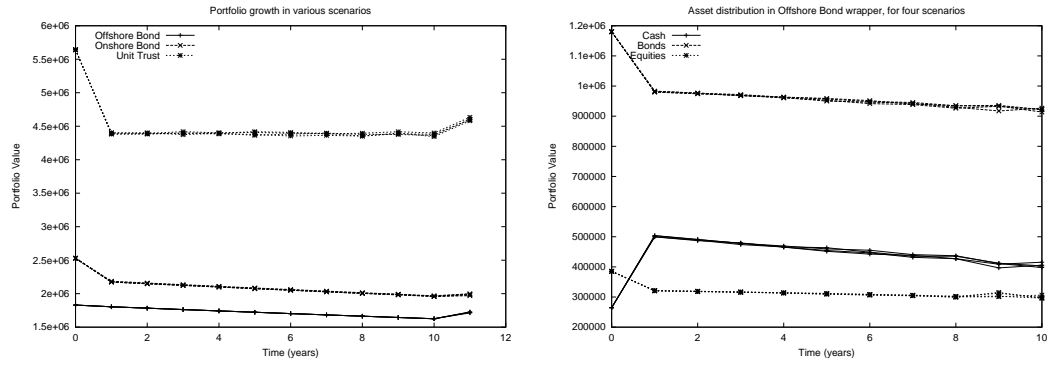
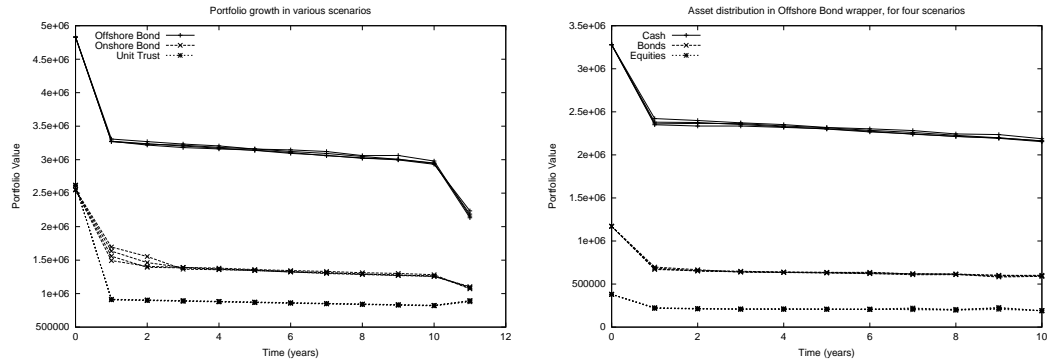


Figure 4: Full taxation case with bounds: wrapper diversification and asset allocation for offshore bond. **Portfolio growth in various scenarios:** Total portfolio growth in three wrappers for the four scenarios

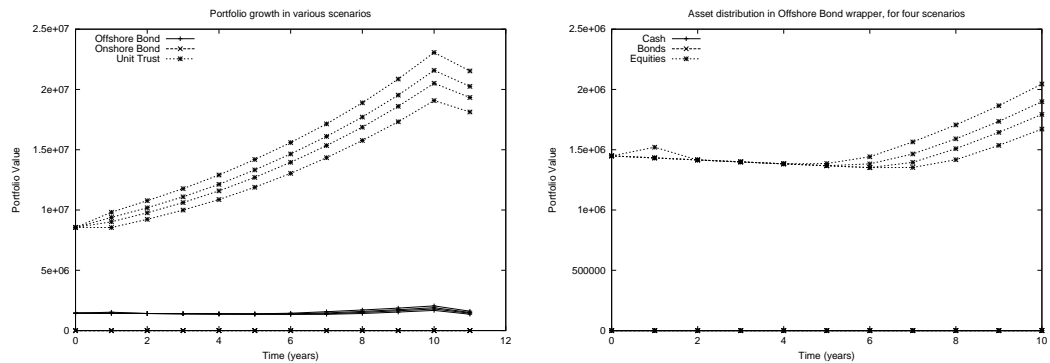
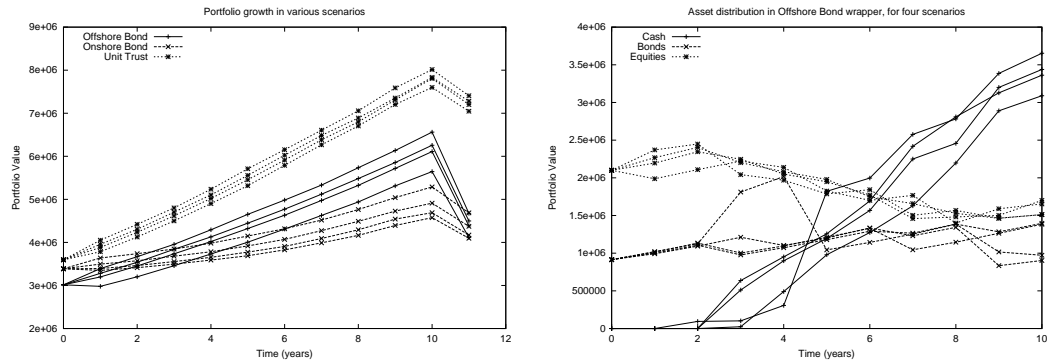
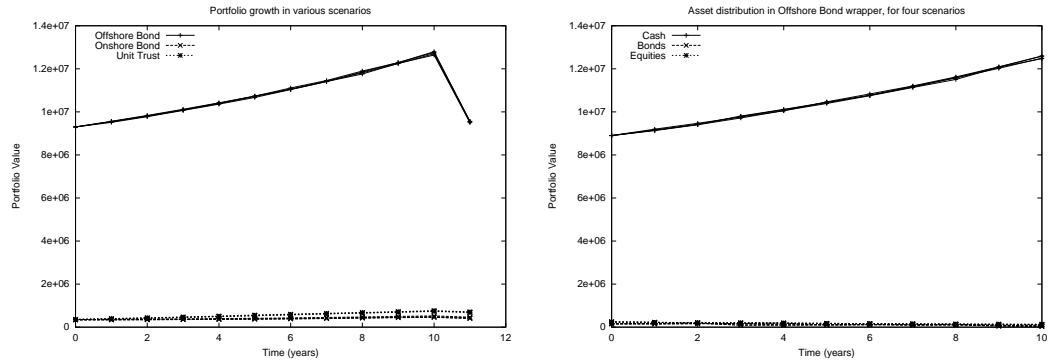
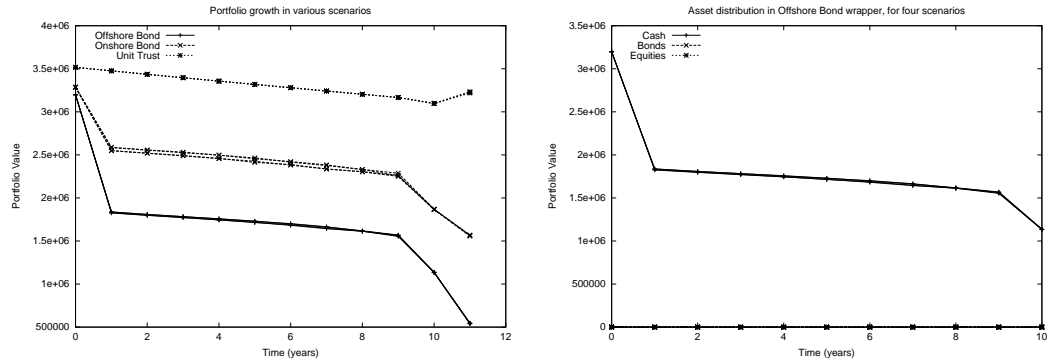
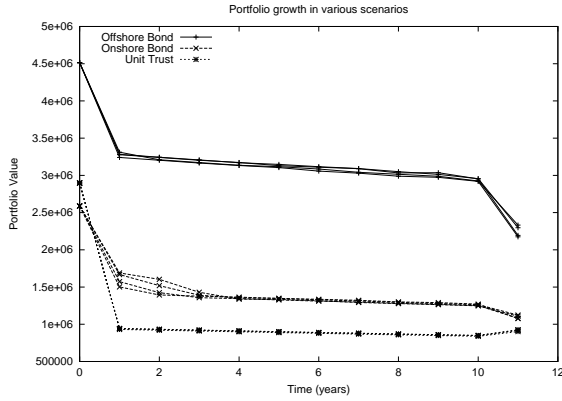
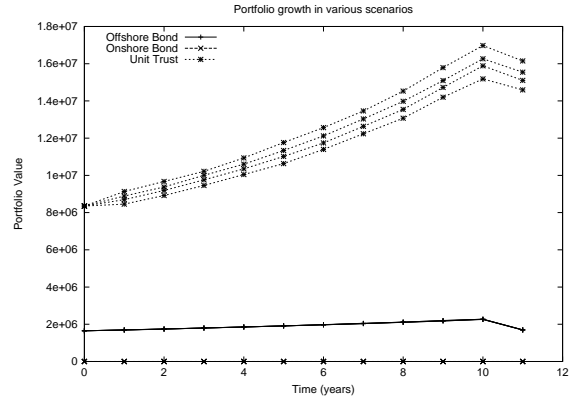


Figure 5: Full taxation case with no bounds: wrapper diversification and asset allocation for offshore bond. **Portfolio growth in various scenarios:** Total portfolio growth in three wrappers for the four scenarios

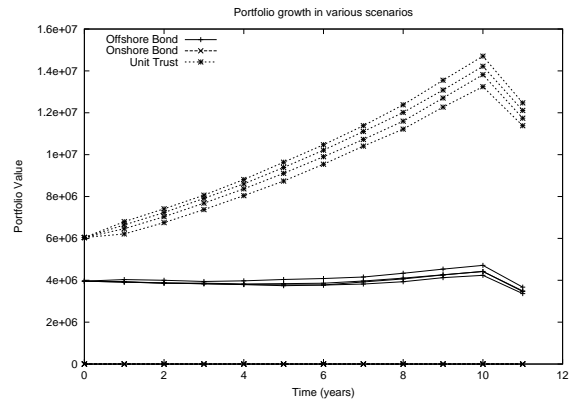
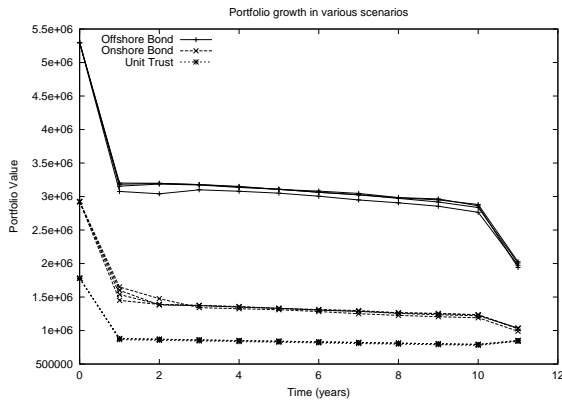
25% risk level



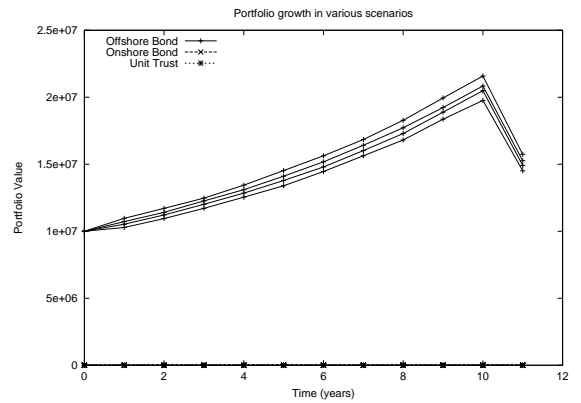
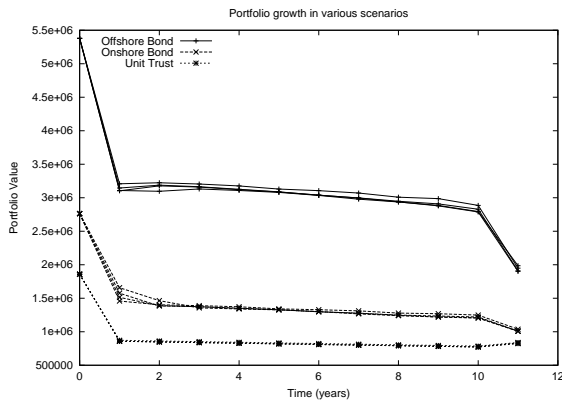
100% risk level



24% Capital Gain Tax



35.45% Capital Gain Tax



40% Capital Gain Tax

Figure 6: Sensitivity Analysis: wrappers allocation for different CGT at 25% and 100% risk. **Portfolio growth in various scenarios:** Total portfolio growth in three wrappers for the four scenarios

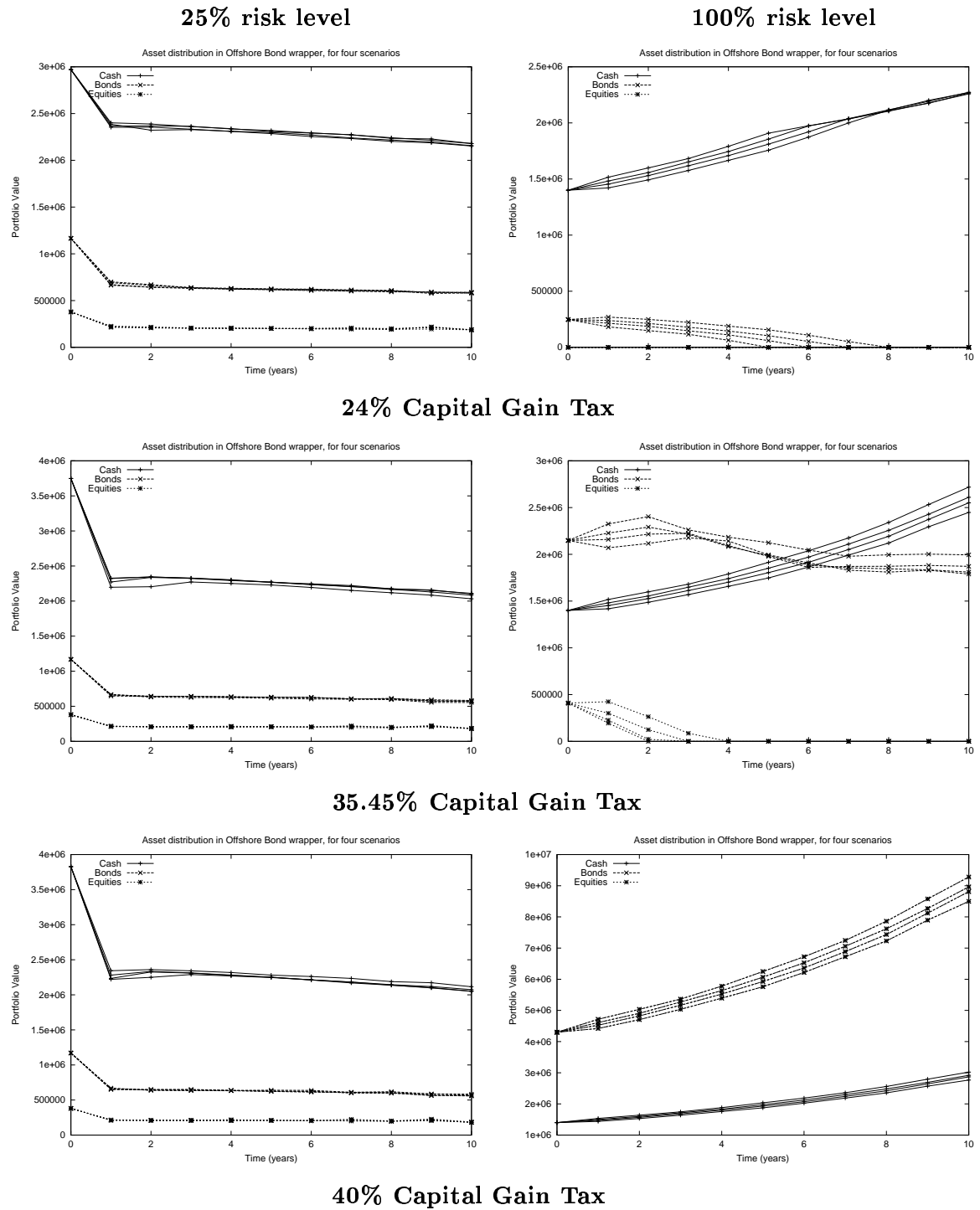


Figure 7: Sensitivity analysis: assets allocation for different CGT at 25% and 100% risk.