Post-Tax Optimization with Stochastic Programming

Maria A. Osorio, Nalan Gülpınar, Reuben Settergren, Berç Rustem

Department of Computing,
Imperial College of Science, Technology and Medicine
180 Queen's Gate, London, SW7 2BZ, UK.

{maosorio,ng2,reuben,br}@doc.ic.ac.uk

September 11, 2001

Abstract

In this paper, we consider a stochastic programming approach to multi-stage post-tax portfolio optimization. Asset performance information is specified as a scenario tree generated by two alternative methods based on simulation and optimization. We assume three tax wrappers involving the same instruments for an efficient investment strategy and determine optimal allocations to different instruments and wrappers. The tax rules are integrated with the linear and mixed integer stochastic models to yield an overall tax and return-efficient multistage portfolio. The computational performance of these models is tested using a case study with different scenario trees.

Keywords Post-tax optimization, portfolio management, multi-stage stochastic programming, mixed integer programming, linear programming, scenario tree generation

1 Introduction

In financial portfolio management, multistage stochastic programming is used to find an optimal investment strategy by maximizing expected wealth subject to constraints specified by the investor [7, 5]. The uncertainty on return values of instruments is represented by a discrete approximation. Given history up to the commencement of the investment period the determination of the finitely many outcomes of the random return variable is called scenario tree generation. Generating scenario trees is important for the performance of the multistage stochastic programming. The root node of the scenario tree represents the decision "today" and the nodes further on represent conditional decisions at later stages. The arcs linking the nodes represent various realizations of the uncertain variables. The dynamics of decision making is thus captured as decisions are adjusted according to realizations of uncertainty.

Taxes have a significant performance impact on portfolio management; often larger than transaction costs for most investors. Therefore, it is important for an investor or a bank to consider the effect of taxes on all investment decisions, including optimal asset allocation. Nevertheless the trade-off between seeking maximum expected return and the tax consequences of investment decisions does not seem to have received sufficient attention. There are well-established methods for pre-tax portfolio management. However, portfolio management after tax seems to have been neglected, possibly due to the complexity of incorporating tax rules within an optimization framework.

In this paper, we consider the post-tax portfolio optimization that takes into account special UK taxation rules for three different wrappers. Post-tax optimization finds an optimal asset allocation with specific tax rules for each asset at each wrapper. We introduce a multistage stochastic approach to the post-tax portfolio management problem. The tax rules are integrated with the linear and mixed integer formulations to yield an overall tax and return-efficient multistage portfolio. The main concern of this paper is to find an optimal investment strategy, after incorparating specific tax rules, using differing asset allocations in wrappers, over a given finite investment horizon. Uncertainty on asset performances (or returns) is represented with a scenario tree. We generate scenario trees by simulation and optimization based approaches [6]. The performance of linear and mixed integer optimization models on different scenario trees is illustrated using a test example.

The rest of the paper is organized as follows. In section 2, we give the terminology and notation for post-tax optimization. Section 3 focuses on the multistage stochastic post-tax linear and mixed integer optimization models. We consider two scenario tree generation methods which are described in Section 4. The computational results measuring the performance of the models are presented in Section 5.

2 Definitions and Notation

In this section, we introduce the financial terminology for multistage post-tax optimization and notation for the mathematical formulation of the problem.

2.1 Scenario Trees

We assume a portfolio of n risky assets in three wrappers and consider its optimal restructuring over a period in terms of expected return after performing tax rules. After the initial investment ($t = 0$), the portfolio may be restructured at discrete times $t = 1, \ldots, T-1$, and redeemed at the end of the period ($t = T$).

Let the increasing σ-field $\mathcal{F}_t$ ($\mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_T$) be generated by stochastic events $\boldsymbol{\rho}^t \equiv \{\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_t\}; t = 1, \ldots, T$. Let the random variables $\mathbf{r}_t(\boldsymbol{\rho}^t)$ and $\mathbf{g}_t(\boldsymbol{\rho}^t)$ denote the uncertain dividend (or income) and capital gain returns on investment. Random variables and some specified coefficients of constraints are assumed to be $\mathcal{F}_t$ measurable functions ($\mathbf{r}_t, \mathbf{g}_t : \Omega_t \to \mathbb{R}^n$) on some probability space $(\Omega_t, \mathcal{F}_t, \mathcal{P}_t)$. Due to the recourse nature of the multistage problem, decision variables $\mathbf{w}_t$, $\mathbf{b}_t$ and $\mathbf{s}_t$ are influenced by previous stochastic events $\boldsymbol{\rho}^t$, and hence $\mathbf{w}_t = \mathbf{w}_t(\boldsymbol{\rho}^t)$, $\mathbf{b}_t = \mathbf{b}_t(\boldsymbol{\rho}^t)$ and $\mathbf{s}_t = \mathbf{s}_t(\boldsymbol{\rho}^t)$. However, for simplicity, we shall use the terms $\mathbf{w}_t$, $\mathbf{b}_t$ and $\mathbf{s}_t$, and assume their implicit dependence on $\boldsymbol{\rho}^t$. We assume that $\boldsymbol{\rho}_t$ can take only finitely many values. Thus, the factors driving the risky events are approximated by a discrete set of scenarios or a sequence of events. Given the event history up to a time t, $\boldsymbol{\rho}^t$, the uncertainty in the next period is characterized by finitely many possible outcomes for the next observation $\boldsymbol{\rho}_{t+1}$. This branching process is represented using a scenario tree.

A scenario is defined as a possible realization of the stochastic variables $\{\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_T\}$. Hence, the set of scenarios corresponds to the set of leaves of the scenario tree, $\mathcal{N}_T$, and nodes of the tree at level $t \geq 1$ (the set $\mathcal{N}_t$) correspond to possible realization of $\boldsymbol{\rho}^t$. We denote a node of the tree (or **event**) by $\mathbf{e} = (s, t)$, where s is a scenario (path from root to leaf), and time period t specifies a particular node on that path. The root of the tree is $\mathbf{0} = (s, 0)$ (where s can be any scenario, since the root node is common to all scenarios). The ancestor (parent) of event $\mathbf{e} = (s, t)$ is denoted $a(\mathbf{e}) = (s, t-1)$, and the branching probability $p_{\mathbf{e}}$ is the conditional probability of event $\mathbf{e}$, given its parent event $a(\mathbf{e})$. The path to event $\mathbf{e}$ is a partial scenario with probability $P_{\mathbf{e}} = \prod p_{\mathbf{e}}$ along that path. Since probabilities $p_{\mathbf{e}}$ must sum to unity at each individual branching, probabilities $P_{\mathbf{e}}$ will sum up to unity across each layer of tree-nodes $\mathcal{N}_t$ for $t = 0, 1, \ldots, T$.

Each node $\mathbf{e} \in \mathcal{N}_t$ at a level $t = 1, \ldots, T$ corresponds to a decision $\{\mathbf{w}_{\mathbf{e}}, \mathbf{b}_{\mathbf{e}}, \mathbf{s}_{\mathbf{e}}\}$ which must be determined at time t, and depends in general on $\boldsymbol{\rho}^t$ and the past decisions $\{\mathbf{w}_j, \mathbf{b}_j, \mathbf{s}_j\}$, for $j = 1, \ldots, t-1$. This process is adapted to $\boldsymbol{\rho}^t$ as $\mathbf{w}_t, \mathbf{b}_t, \mathbf{s}_t$ cannot depend on future events $\boldsymbol{\rho}_{t+1}, \ldots, \boldsymbol{\rho}_T$ which are not yet realized.

2.2 Investment Wrappers and Tax Rules

A wrapper is a set of assets, such as equities, bond, cash and properties, with a specific set of rules for taxation on regular basis. Different investor's life events such as withdrawals, gifts, or emigration [13] affect taxation rules and the performance of the portfolio at the end of the investment horizon. Three different wrappers, namely offshore bonds, onshore bonds and unit trust are considered. Returns are obtained as dividends, income, and capital gains from instruments within each wrapper.

Taxes are payable in different ways for different wrappers and assets and receive special treatment in specific situations according to the wrapper utilized. The main taxes applicable in this paper are income, capital gains, tax within bond and tax on gross returns. In addition, we assume that an investor is allowed to withdraw some amount of money from different wrappers with different tax penalties.

There is no annual tax for the offshore bond wrapper; but at the time of encashment, tax on the gross returns (whether from capital gain or income) is due at a rate of t_{gb}. The onshore bond wrapper is very similar to offshore bond, except that some tax is due every year (t_{gr}), and tax is also due at encashment (t_{ob}). While dividend or income returns from investments in a unit trust wrapper are subject to annual income tax, t_{inc}, capital gain taxes are deferred until encashment.

Profits realised on the disposal of certain types of assets are subject to capital gains tax. These include direct holdings of shares and property, as well as unit trusts. However, there is some mitigation of this tax in the form of taper relief. Taper relief is a gradual reduction of the amount of tax payable, dependent on the length of time an asset has been held. Taper relief is provided, so that the longer the time horizon until encashment, the lower the tax.

2.3 Notation

Our notation is described in Table 2.3. All quantities in boldface represent vectors in $\mathbb{R}^n$. The transpose of a vector is denoted with the symbol $'$. In Table 2.3, subscript $*$ indicates that vectors have two indices. The first index represents wrappers $k = 1, 2, 3$ for offshore bonds, onshore bonds and unit trust, respectively. The second one denotes each event $\mathbf{e} \in \mathcal{N}_t$ at time $t = 1, \ldots, T$ of the scenario tree.

	Symbols and Input Data
$\mathbf{1}$	$\equiv (1, 1, 1, \ldots, 1)'$
$\mathbf{p} \circ \mathbf{q}$	$\equiv (p_1 q_1,\ p_2 q_2,\ \ldots, p_n q_n)'$ (Hadamard product)
$\mathbf{p}' \mathbf{q}$	$\equiv p_1 q_1 + p_2 q_2 + \ \ldots\ + p_n q_n$ (Inner product)
$\mathbf{e} \equiv (s, t)$	index denoting an event (a node of the scenario tree)
$a(\mathbf{e})$	ancestor of event $\mathbf{e}$ (parent in the scenario tree)
$\mathcal{N}_t$	set of nodes of the scenario tree at time t
$p_{\mathbf{e}}$	branching probability of event $\mathbf{e}$: $p_{\mathbf{e}} = \text{Prob}[\mathbf{e} \mid a(\mathbf{e})]$
$P_{\mathbf{e}}$	probability of event $\mathbf{e}$: if $\mathbf{e} = (s, t)$, then $P_{\mathbf{e}} = \prod_{i=1 \ldots t} p_{(s,i)}$
n	number of investment assets
M	amount of initial investment
T	investment planning horizon
TW_t	total withdrawal at time t
ic_k	percentage paid in initial cost for wrapper k
ac_k	percentage paid in annual cost for wrapper k
$\mathbf{r}_{k\mathbf{e}}$	dividends or income returns for wrapper k at node $\mathbf{e}$
$\mathbf{g}_{k\mathbf{e}}$	capital gains for wrapper k at node $\mathbf{e}$
t_{gb}	tax on gross returns of offshore bond
t_{ob}	tax within onshore bond
t_{gr}	tax at the end on gross returns from onshore bond
$\mathbf{t_{in}}$	income tax paid on dividends or income
CG_t	capital gains tax at period t
tc	transaction cost
	Decision Variables
R_k	cumulative returns for wrapper k
NR_k	net redemption value obtained from wrapper k
CT_*	accumulated tax
$\mathbf{w}_*$	amount of money held in each asset
$\mathbf{b}_*$	amount bought of each asset
$\mathbf{s}_*$	amount sold of each asset
$\mathbf{h}_*$	first withdrawal from a wrapper
$\mathbf{u}_*$	excess withdrawal from a wrapper
$\mathbf{f}_*$	withdrawal taken from the original investment in a wrapper
$\mathbf{y}_{kt}$	binary variable for wrapper k at t

Table 1. Notation

3 Multistage Post-Tax Optimization Model

In this section, we present multistage post-tax optimization model, which describes the way that the initial investment is allocated, the way money grows and can be transferred between assets at each time period, the way diversification is enforced, returns accumulate, and withdrawals can be taken. Finally, the LP and MIP models finding efficient post-tax investment allocations are described

3.1 Constraints

Initial Allocation

At $t = 0$, the initial investment is distributed among instruments within each wrappers such that the following constraint is satisfied.

$$\sum_{k=1}^{3} \mathbf{1}'\mathbf{w}_{k0} = M \tag{1}$$

Cash Balance Equations

Subsequent transactions do not alter the wealth within the period in question. Therefore, the following constraints specify to balance the portfolio for each wrapper $k = 1, 2, 3$ at node $\mathbf{e} \in \mathcal{N}_t$ for $t = 1, \ldots, T$

$$\mathbf{1}'\mathbf{b}_{k\mathbf{e}} - \mathbf{1}'\mathbf{s}_{k\mathbf{e}} = 0 \tag{2}$$

These constraints basically balance the cash by reallocating money among assets within each wrapper. In this way, sales fund the purchase of other assets in the same wrapper.

Wealth

The wealth for each wrapper is described by the wealth of the previous year plus the increased value of the assets in that wrapper after paying the corresponding annual taxes. The transaction also allows selling or buying assets within the same wrappers, with corresponding transaction costs. Therefore, the wealth for each asset within wrappers $k = 1, 2, 3$ at node $\mathbf{e} \in \mathcal{N}_t$ $t = 1, \ldots, T$ of the scenario tree are as follows;

$$\begin{aligned}
\mathbf{w}_{1\mathbf{e}} &= (1 - ac_1)\left[(\mathbf{1} + \mathbf{r}_\mathbf{e} + \mathbf{g}_\mathbf{e}) \circ \mathbf{w}_{1a(\mathbf{e})}\right] + (1 - tc)\mathbf{b}_{1\mathbf{e}} - \mathbf{s}_{1\mathbf{e}} \\
\mathbf{w}_{2\mathbf{e}} &= (1 - ac_2)\left[(\mathbf{1} + (1 - t_{ob})(\mathbf{r}_\mathbf{e} + \mathbf{g}_\mathbf{e})) \circ \mathbf{w}_{2a(\mathbf{e})}\right] + (1 - tc)\mathbf{b}_{2\mathbf{e}} - \mathbf{s}_{2\mathbf{e}} \\
\mathbf{w}_{3\mathbf{e}} &= (1 - ac_3)\left[(\mathbf{1} + (\mathbf{1} - \mathbf{t}_{\mathbf{in}}) \circ \mathbf{r}_\mathbf{e} + \mathbf{g}_\mathbf{e}) \circ \mathbf{w}_{3a(\mathbf{e})}\right] + (1 - tc)\mathbf{b}_{3\mathbf{e}} - \mathbf{s}_{3\mathbf{e}}
\end{aligned} \tag{3}$$

Notice that the annual bank fees deducted by term $(1 - ac_k)$ for $k = 1, 2, 3$ must be augmented by the bank's initial setup fees in the first year. For children of the root scenario node, $\mathbf{e} \in \mathcal{N}_1$, the term becomes $(1 - ic_k - ac_k)$, and is imposed on all constraints.

Cumulative Taxes

While offshore and onshore bonds accumulate taxes on overall returns, unit trust accumulates taxes only on the capital gains of the investment. Cumulative tax is paid at the end of investment horizon T when the investment is encashed. At beginning of the investment plan (that is, at the root node of scenario tree), the cumulative tax for each wrapper is assumed to be zero

$$CT_{\mathbf{10}} = CT_{\mathbf{20}} = CT_{\mathbf{30}} = 0 \tag{4}$$

In time periods $t = 1, \ldots, T$ for each node of the scenario tree $\mathbf{e} \in \mathcal{N}_t$, cumulative taxes for wrappers $k = 1, 2, 3$ are calculated by the following equations

$$
\begin{aligned}
CT_{1\mathbf{e}} &= CT_{1a(\mathbf{e})} + t_{gb}(1-ac_1)\left[(\mathbf{r_e}+\mathbf{g_e})'\mathbf{w}_{1\mathbf{e}}\right] \\
CT_{2\mathbf{e}} &= CT_{2a(\mathbf{e})} + t_{gr}(1-ac_2)\left[(\mathbf{r_e}+\mathbf{g_e})'\mathbf{w}_{2\mathbf{e}}\right] \\
CT_{3\mathbf{e}} &= CT_{3a(\mathbf{e})} + CG_T(1-ac_3)\left[\mathbf{g_e}'\mathbf{w}_{3\mathbf{e}}\right]
\end{aligned} \tag{5}
$$

Cumulative Returns

Assume that returns $R_{k\mathbf{e}}$ obtained from wrappers $k = 1,2,3$ at each node $\mathbf{e} \in \mathcal{N}_t$ for $t = 1,\ldots,T$ of the scenario tree are nonnegative, $R_{k\mathbf{e}} \geq 0$, and are initialized as zero at the root node of the scenario tree; that is $R_{1\mathbf{0}} = R_{2\mathbf{0}} = R_{3\mathbf{0}} = 0$.

In offshore and onshore bonds, the returns are accumulated every year. However, in the unit trust, returns include the income or dividend after income tax and the capital gain tax are deducted. For all $\mathbf{e} \in \mathcal{N}_t$, $t = 1,\ldots,T$, the returns obtained by each assets within each wrapper are calculated as

$$
\begin{aligned}
R_{1\mathbf{e}} &= R_{1a(\mathbf{e})} + (1-ac_1)\left[(\mathbf{r_e}+\mathbf{g_e})'\mathbf{w}_{1\mathbf{e}}\right] \\
R_{2\mathbf{e}} &= R_{2a(\mathbf{e})} + (1-ac_2)(1-t_{ob})\left[(\mathbf{r_e}+\mathbf{g_e})'\mathbf{w}_{2\mathbf{e}}\right] \\
R_{3\mathbf{e}} &= (1-ac_3)\left[(\mathbf{1}-\mathbf{t_{in}})\circ\mathbf{r_e}+\mathbf{g_e}\right]'\mathbf{w}_{3\mathbf{e}}
\end{aligned} \tag{6}
$$

We note that $\mathbf{t_{in}}$ is a vector whose elements correspond to different income tax rates for different assets within the wrapper.

Diversification Constraints

Any diversification restriction imposed by the investor or the bank's advice can be expressed by the percentage upper bounds for each asset $i = 1,\ldots,n$ within each wrapper at node $\mathbf{e} \in \mathcal{N}_t$ $(t = 1,\ldots,T)$ of event tree as

$$
\sum_{k=1}^{3} w_{ki\mathbf{e}} \leq w_{ki\mathbf{e}}^{u} \sum_{k=1}^{3} (\mathbf{1}'\mathbf{w}_{k\mathbf{e}}) \tag{7}
$$

where $w_{ki\mathbf{e}}^{u}$ represents an upper bound for asset i.

3.2 Multistage LP Problem

The Linear Programming (LP) model maximizes the expected wealth at the end of the investment horizon after applying specific tax rules for differing asset allocations in wrappers. Expected wealth is calculated as the total net redemption value for each wrappers at time period T.

The redemption value is basically defined as the net amount of money received at time T when a wrapper is encashed after taxes are paid. The constraints of LP model express the wealth, return, cash balance and structure of tax rules. The general form of the LP model can be stated as follows:

$$
\begin{array}{ll}
\max \sum_{k=1}^{3} NR_k & \\
\text{subject to} & \\
\quad \text{Constraints } (1), (2), \ldots, (7) & \\
\quad NR_k = \sum_{\mathbf{e} \in \mathcal{N}_T} P_{\mathbf{e}} \left[\mathbf{1}'(\mathbf{w}_{k\mathbf{e}} - CT_{k\mathbf{e}}) \right] & k = 1,2,3 \\
\quad NR_k \geq 0 & k = 1,2,3 \\
\quad CT_{k\mathbf{e}}, R_{k\mathbf{e}} \geq 0 & \mathbf{e} \in \mathcal{N}_t,\ t = 1, \ldots, T,\ k = 1,2,3 \\
\quad \mathbf{w}_{k\mathbf{e}}, \mathbf{b}_{k\mathbf{e}}, \mathbf{s}_{k\mathbf{e}} \geq 0 & \mathbf{e} \in \mathcal{N}_t,\ t = 1, \ldots, T,\ k = 1,2,3 \\
\quad R_{1\mathbf{0}} = R_{2\mathbf{0}} = R_{3\mathbf{0}} = 0 &
\end{array}
$$

The number of variables and constraints in the LP model is increased by the number of assets, wrappers and the topology of the scenario tree. The size of the scenario tree depends on the depth and branching at each time period. Our computational results show that even for large scenario trees, it is possible to find optimal investment strategies with the LP model in a reasonable amount of CPU time.

3.3 Withdrawals

Life events play an important role in activating the taxation rules in the LP model. In this paper, we only consider withdrawals; for other events such as gifts, emigration, death and inheritance, the reader is referred to [13]. If an investor is allowed to withdraw some amount of money at any time period of investment horizon, the objective function of LP is then a maximization of future wealth after tax is applied, according to the investor's withdrawal.

For withdrawals, the net growth obtained from each asset within wrappers is preferred to the capital gains since it is tax deferred until the end of the horizon. Therefore, withdrawals are allowed to be taken from the income, dividends or capital gains in unit trust, and from returns in onshore and offshore bonds in any year subject to the following restrictions.

In offshore bond wrapper,

- withdrawals up to 5% of the original investment per year may be taken, and the taxes deferred until encashment. Unused portions of the annual allowance of 5% may be carried forward, i.e. if no withdrawals are taken until year 5, then tax-deferred withdrawals may be as large as 25% of the original investment may be taken.
- withdrawals beyond the cumulative 5% limit may be taken, and are subject to immediate tax at the encashment rate of t_{gb}.
- withdrawals may only be taken from the gains since investment — original capital must remain in the wrapper until encashment.

In onshore bond wrapper, rules for withdrawals are the same as offshore bond, noting that tax-liable withdrawals are taxed at the onshore bond encashment rate of t_{ob}.

Withdrawals from unit trust may be taken from the current year's growth only; any returns not used for withdrawals are rolled back into the value of the unit trust. As dividend and income returns have their taxes paid annually, withdrawals can be taken from them without further tax. Withdrawals from growth due to capital gains, however, are subject to tax at the tapered rate of the year of the withdrawal.

In this case, the constraints (3), (5) and (6) formulated in the previous section are replaced by the following constraints reflecting different tax rules for wealths, cumulative taxes and returns by taking into account the effect of the withdrawals as follows.

The wealth gained at node $\mathbf{e} \in \mathcal{N}_t$ for $t = 1, \ldots, T$ becomes

$$\begin{aligned}
\mathbf{w}_{1\mathbf{e}} &= (1 - ac_1)\left[(\mathbf{1} + \mathbf{r_e} + \mathbf{g_e}) \circ \mathbf{w}_{1a(\mathbf{e})}\right] - \mathbf{h}_{1\mathbf{e}} - \frac{1}{1 - t_{gb}}\mathbf{u}_{1\mathbf{e}} + (1 - tc)\mathbf{b}_{1\mathbf{e}} - \mathbf{s}_{1\mathbf{e}} \\
\mathbf{w}_{2\mathbf{e}} &= (1 - ac_2)\left[(\mathbf{1} + (1 - t_{ob})(\mathbf{r_e} + \mathbf{g_e})) \circ \mathbf{w}_{2a(\mathbf{e})}\right] - \mathbf{h}_{2\mathbf{e}} - \frac{1}{1 - t_{gr}}\mathbf{u}_{2\mathbf{e}} \\
&\quad + (1 - tc)\mathbf{b}_{2\mathbf{e}} - \mathbf{s}_{2\mathbf{e}} \\
\mathbf{w}_{3\mathbf{e}} &= (1 - ac_3)\left[(\mathbf{1} + (\mathbf{1} - \mathbf{t_{in}}) \circ \mathbf{r_e} + \mathbf{g_e}) \circ \mathbf{w}_{3a(\mathbf{e})}\right] - \mathbf{h}_{3\mathbf{e}} - \frac{1}{1 - CG_\mathbf{e}}\mathbf{u}_{3\mathbf{e}} \\
&\quad + (1 - tc)\mathbf{b}_{3\mathbf{e}} - \mathbf{s}_{3\mathbf{e}}
\end{aligned} \tag{8}$$

In (8), the variable $\mathbf{h}_{k\mathbf{e}}$ represents the part of withdrawal taken from the cumulative 5% of the original investment in offshore and onshore bonds or dividend (after income tax) in unit trust. This amount is tax deferred at time t. The variable $\mathbf{u}_{k\mathbf{e}}$ describes the part of withdrawal taken from overall returns that exceed the 5% cumulative of the original investment in offshore and onshore bonds, or comes from capital gains in unit trust at that year. In this case, the amount is subject to tax at the rate of encashment in onshore and offshore bonds and at the rate of capital tax at that year in unit trust. The coefficients of variable $\mathbf{u}$, in the form of $\frac{1}{1-x}$, represent withdrawals subject to immediate tax. If x is the tax rate applied to the withdrawal the coefficient ensures that an appropriately larger withdrawal is removed from the investment, so that the desired amount ($\mathbf{u}$) remains after tax is paid.

Deferred tax is accumulated from the root node to each leaf node of the scenario tree, to be paid when the corresponding wrapper is encashed at the last time period. In this case, the taxes paid on excess withdrawals are taken into account. The cumulative tax constraints are stated as follows;

$$\begin{aligned}
CT_{1\mathbf{e}} &= CT_{1a(\mathbf{e})} + t_{gb}(1 - ac_1)\left[(\mathbf{r_e} + \mathbf{g_e})^{'}\mathbf{w}_{1\mathbf{e}}\right] - \frac{t_{gb}}{1 - t_{gb}}\mathbf{1}^{'}\mathbf{u}_{1\mathbf{e}} \\
CT_{2\mathbf{e}} &= CT_{2a(\mathbf{e})} + t_{gr}(1 - ac_2)\left[(\mathbf{r_e} + \mathbf{g_e})^{'}\mathbf{w}_{2\mathbf{e}}\right] - \frac{t_{gr}}{1 - t_{gr}}\mathbf{1}^{'}\mathbf{u}_{2\mathbf{e}} \\
CT_{3\mathbf{e}} &= CT_{3a(\mathbf{e})} + CG_T(1 - ac_3)\left[\mathbf{g}_\mathbf{e}^{'}\mathbf{w}_{3\mathbf{e}}\right] - \frac{CG_T}{1 - CG_\mathbf{e}}\mathbf{u}_{3\mathbf{e}}
\end{aligned} \tag{9}$$

For all $\mathbf{e} \in \mathcal{N}_t,\ t = 1, \ldots, T$, the returns obtained from each wrapper are calculated by considering the first and excess withdrawals as follows

$$\begin{aligned}
R_{1\mathbf{e}} &= R_{1a(\mathbf{e})} + (1 - ac_1)\left[(\mathbf{r_e} + \mathbf{g_e})^{'}\mathbf{w}_{1\mathbf{e}}\right] - \mathbf{1}^{'}\mathbf{h}_{1\mathbf{e}} - \frac{1}{1 - t_{gb}}\mathbf{1}^{'}\mathbf{u}_{1\mathbf{e}} \\
R_{2\mathbf{e}} &= R_{2a(\mathbf{e})} + (1 - ac_2)(1 - t_{ob})\left[(\mathbf{r_e} + \mathbf{g_e})^{'}\mathbf{w}_{2\mathbf{e}}\right] - \mathbf{1}^{'}\mathbf{h}_{2\mathbf{e}} - \frac{1}{1 - t_{gr}}\mathbf{1}^{'}\mathbf{u}_{2\mathbf{e}} \\
R_{3\mathbf{e}} &= (1 - ac_3)\left[(\mathbf{1} - \mathbf{t_{in}}) \circ \mathbf{r_e} + \mathbf{g_e}\right]^{'}\mathbf{w}_{3\mathbf{e}} - \mathbf{1}^{'}\mathbf{h}_{3\mathbf{e}} - \frac{1}{1 - CG_\mathbf{e}}\mathbf{1}^{'}\mathbf{u}_{3\mathbf{e}}
\end{aligned} \tag{10}$$

where $R_{k\mathbf{e}} \geq 0$ so that the variables $\mathbf{h}_{k\mathbf{e}}, \mathbf{u}_{k\mathbf{e}}$ are prevented from withdrawing money from the capital.

In addition, the following constraints describing the structure of the withdrawals need to be added to the LP model. The total withdrawal at $\mathbf{e} \in \mathcal{N}_t$ for $t = 1, \ldots, T$ from wrappers

$k = 1, 2, 3$ is computed by

$$TW_{\mathbf{e}} = \sum_{k=1}^{3} \left[\mathbf{1}^{'} \mathbf{h}_{ke} + \mathbf{1}^{'} \mathbf{u}_{k\mathbf{e}} \right], \quad \forall \mathbf{e} \in \mathcal{N}_t, \ t = 1, \ldots, T \tag{11}$$

In this case, it is ensured that the money taken from every wrapper and asset at any time period will be enough to cover the total amount of withdrawal.

At any time period t, the first withdrawal from onshore and offshore bonds must not be exceed 5% of the initial investment. In order to build the constraints limiting tax deferred withdrawals to the cumulative 5%, we define the sum from $\mathbf{e}' = 0$ to $\mathbf{e}' = \mathbf{e}$ to range over the partial scenario path from root of the event $\mathbf{e}$.

$$\sum_{\mathbf{e}'=0}^{\mathbf{e}} \mathbf{1}^{'} \mathbf{h}_{k\mathbf{e}'} \leq (0.05t) \left[\mathbf{1}^{'} \mathbf{w}_{k\mathbf{0}} \right], \quad \forall \, \mathbf{e} \in \mathcal{N}_t \ \ t = 1, \ldots, T \ \ k = 1, 2 \tag{12}$$

The unit trust requires only the first withdrawal to be limited to year t growth such that

$$\mathbf{h}_{3\mathbf{e}} \leq (1 - ac_3) \left[(\mathbf{1} - \mathbf{t}_{\mathbf{in}}) \circ \left(\mathbf{r}_{\mathbf{e}} \circ \mathbf{w}_{3a(\mathbf{e})} \right) \right], \quad \forall \mathbf{e} \in \mathcal{N}_t, \ t = 1, \ldots, T \tag{13}$$

At each node $\mathbf{e} \in \mathcal{N}_t, t = 1, \ldots, T$ of the scenario tree for wrappers $k = 1, 2, 3$, the excess withdrawals are bounded above such that the following inequalities are satisfied.

$$\mathbf{u}_{3\mathbf{e}} \leq (1 - ac_3)(1 - CG_{\mathbf{e}}) \left[\mathbf{g}_{\mathbf{e}} \circ \mathbf{w}_{3a(\mathbf{e})} \right] \tag{14}$$

3.4 Multistage MIP Problem

In the LP model, it is assumed that withdrawals do not exceed the total returns for each instrument within each wrapper at any time period. Although an investor usually prefers or is advised to take money obtained from the income or the dividend rather than the original capital, the net growth does not always cover the amount of withdrawal specified. In this case, the original capital needs to be taken into consideration. The withdrawal is composed of funds obtained from the income or dividends, the capital gains and the original capital. So an optimization model would naturally attempt to take withdrawal from capital rather than paying immediate taxes. Therefore, LP is extended to the multistage mixed integer (MIP) model by introducing binary variables.

If money is withdrawn from the income or dividends in unit trust or 5% cumulative annual in offshore and onshore bonds, then it is tax deferred. However, if money is from capital gains in unit trust, then capital gain tax has to be paid. If money is withdrawn from the original capital, it is tax-free for all wrappers. In order to model this special case, we need to add a variable for each withdrawal taken from the original investment $\mathbf{f}_k$ and a binary variable $y_{k\mathbf{e}}$ at node $\mathbf{e} \in \mathcal{N}_t$ for each wrapper k. The binary variables represent the use of the original capital, only when all money from returns or capital gains in each wrapper is already taken.

The following constraints allow money for withdrawals to be taken from the original capital $\mathbf{f}_{k\mathbf{e}}$ when $y_{k\mathbf{e}} = 1$. However, when $y_{k\mathbf{e}} = 0$, the constraints force withdrawals to use only net returns through $\mathbf{h}_{k\mathbf{e}}$ and $\mathbf{u}_{k\mathbf{e}}$.

$$\mathbf{1}^{'} \mathbf{f}_{k\mathbf{e}} - M y_{k\mathbf{e}} \leq 0, \quad \forall \, k = 1, 2, 3 \ \ \mathbf{e} \in \mathcal{N}_t \ \ t = 1, \ldots, T \tag{15}$$

$$R_{k\mathbf{e}} + M y_{k\mathbf{e}} \leq M, \quad \forall \, k = 1, 2, 3 \ \ \mathbf{e} \in \mathcal{N}_t \ \ t = 1, \ldots, T \tag{16}$$

The total amount of withdrawal at $\mathbf{e} \in \mathcal{N}_t$ for $t = 1, \dots, T$ is extended to include all types of withdrawals as well as the one taken from the original investment. It is computed as

$$TW_{\mathbf{e}} = \sum_{k=1}^{3} \left[\mathbf{1}^{'}\mathbf{h}_{k\mathbf{e}} + \mathbf{1}^{'}\mathbf{u}_{k\mathbf{e}} + \mathbf{1}^{'}\mathbf{f}_{k\mathbf{e}} \right] \tag{17}$$

The withdrawal from the original investment is cooperated such a way that $\mathbf{w}_{k\mathbf{e}}$ at node $\mathbf{e} \in \mathcal{N}_t$ for $t = 1, \dots, T$ are as follows;

$$\begin{aligned}
\mathbf{w}_{1\mathbf{e}} &= (1 - ac_1)\left[(\mathbf{1} + \mathbf{r}_{\mathbf{e}} + \mathbf{g}_{\mathbf{e}}) \circ \mathbf{w}_{1a(\mathbf{e})}\right] - \mathbf{h}_{1\mathbf{e}} - \frac{1}{1 - t_{gb}}\mathbf{u}_{1\mathbf{e}} \\
&\quad - \mathbf{f}_{1\mathbf{e}} + (1 - tc)\mathbf{b}_{1\mathbf{e}} - \mathbf{s}_{1\mathbf{e}} \\
\mathbf{w}_{2\mathbf{e}} &= (1 - ac_2)\left[(\mathbf{1} + (1 - t_{ob})(\mathbf{r}_{\mathbf{e}} + \mathbf{g}_{\mathbf{e}})) \circ \mathbf{w}_{2a(\mathbf{e})}\right] - \mathbf{h}_{2\mathbf{e}} - \frac{1}{1 - t_{gr}}\mathbf{u}_{2\mathbf{e}} \\
&\quad - \mathbf{f}_{2\mathbf{e}} + (1 - tc)\mathbf{b}_{2\mathbf{e}} - \mathbf{s}_{2\mathbf{e}} \\
\mathbf{w}_{3\mathbf{e}} &= (1 - ac_3)\left[(\mathbf{1} + (\mathbf{1} - \mathbf{t}_{\mathbf{in}}) \circ \mathbf{r}_{\mathbf{e}} + \mathbf{g}_{\mathbf{e}}) \circ \mathbf{w}_{3a(\mathbf{e})}\right] - \mathbf{h}_{3\mathbf{e}} - \frac{1}{1 - CG_{\mathbf{e}}}\mathbf{u}_{3\mathbf{e}} \\
&\quad - \mathbf{f}_{3\mathbf{e}} + (1 - tc)\mathbf{b}_{3\mathbf{e}} - \mathbf{s}_{3\mathbf{e}}
\end{aligned} \tag{18}$$

The MIP model is then stated as follows;

$$\begin{aligned}
&\max \sum_{k=1}^{3} NR_k \\
&\text{subject to} \\
&\qquad \text{Constraints (1), (2), (4), (7), (9), (10)} \\
&\qquad \text{Constraints (12), \dots, (18)} \\
&\qquad NR_k = \sum_{\mathbf{e} \in \mathcal{N}_T} P_{\mathbf{e}} \left[\mathbf{1}'(\mathbf{w}_{k\mathbf{e}} - CT_{k\mathbf{e}})\right] && k = 1, 2, 3 \\
&\qquad NR_k \geq 0 && k = 1, 2, 3 \\
&\qquad CT_{k\mathbf{e}}, R_{k\mathbf{e}} \geq 0 && k = 1, 2, 3, \ \ \mathbf{e} \in \mathcal{N}_t, \ \ t = 1, \dots, T \\
&\qquad \mathbf{w}_{k\mathbf{e}}, \mathbf{b}_{k\mathbf{e}}, \mathbf{s}_{k\mathbf{e}} \geq \mathbf{0} && k = 1, 2, 3, \ \ \mathbf{e} \in \mathcal{N}_t, \ \ t = 1, \dots, T \\
&\qquad y_{k\mathbf{e}} \in \{0, 1\} && k = 1, 2, 3, \ \ \mathbf{e} \in \mathcal{N}_t, \ \ t = 1, \dots, T \\
&\qquad R_{1\mathbf{0}} = R_{2\mathbf{0}} = R_{3\mathbf{0}} = 0
\end{aligned}$$

Notice that when all binary variables are $y_{k\mathbf{e}} = 0$, the MIP problem is then the same as the LP problem. It is well known in the mathematical programming community that to solve MIP problems is computationally challenging when the number of integer variables increases. The number of binary variables, of course, depends on the size of the scenario tree as well as the number of withdrawals required. Despite the size disadvantage of the MIP model, it is a more realistic approach to the post-tax optimization problem considered in this paper, as can be seen from our computational results.

4 Scenario Tree Generation

Multistage stochastic programming requires a coherent representation of uncertainty. This is expressed in terms of multivariate continuous distributions. Hence, a decision model is generated with internal sampling or a discrete approximation of the underlying continuous distribution. For the post-tax optimization model, the random variables are the uncertain return values of each asset on an investment. The discretization of the random values and the probability space leads to a framework in which a random variable takes finitely many values. At each time period, new scenarios branch from the old, creating a scenario tree. In this paper, we generate scenario trees using two approaches based on probabilistic simulation and optimization methods. For further details of these procedures as well as alternative methods for scenario tree generation, the reader is referred to [6].

In the simulation based scenario tree generation approach, the price scenarios at each time period are generated as the centroids of simulations generated in parallel or sequentially. One time period of growth from "today" to "tomorrow" in each scenario is simulated. A large number of randomly generated simulations is then clustered into groups. If k branches are desired from the current scenario tree node, then k clusters need to be formed. Initially, the seed points around which the clusters are built might as well be chosen to be the first k scenarios, since the scenarios are independently generated, and are in arbitrary order. If the resulting clustering fails to meet the criteria applied in the test stage, new seed points will have to be chosen, and the clustering process repeated, until the criteria is met. The distance measure to determine which seed each scenario is the closest to can be chosen with great flexibility as well as the way the simulations generated.

In the optimization based approach to generate a scenario tree, the market expectations are specified by the statistical properties that are relevant to the problem considered. The central moments and co-moments are the statistical properties of the multivariate distribution. The event tree is constructed so that these statistical properties are matched. This is done by letting stochastic returns and probabilities in the scenario tree be the variables in a nonlinear optimization problem. The objective is to minimize the weighted square distance between the statistical properties specified by the decision maker and the statistical properties of the constructed tree. The tree is either generated node by node; that is solving a small size nonlinear optimization problem at each node of the event tree or whole tree is obtained by solving a large optimization problem. For this specific example, the first four central moments: mean, standard deviation, skewness, and kurtosis, and covariances as co-moments, are matched.

The input to the simulation based scenario tree generation procedure consists of the structure of the scenario tree, the number of random simulations to generate, and statistical parameters for the simulation of scenarios. For the optimization based approach, the topology of the scenario tree as well as the statistical properties have to be provided. For the data used in section 5, the statistical parameters were measured from historical data: 151 monthly valuations of cash, bonds, and equities in the UK market, from 1988 to mid-2000.

Data for each of the three assets (cash, bonds, and equities) was fit to an exponential growth curve. The obtained monthly growth rates were annualized, and used to simulate future growth. A covariance matrix and the central moments were measured from the residuals of the exponential fit, and used in the optimization based scenario generation methods. Cash is assumed to be risk-free and therefore its corresponding variance, skewness and kurtosis are initialized as zero.

5 Computational Results

5.1 Implementation

A dedicated solver was built in C++ for the LP model described above, using the interior-point linear/quadratic solver BPMPD (Fortran) [12] to optimize the generated models. This code was an outgrowth of the multistage portfolio analysis program *foliage* [7], in which taxes are not modeled. For the MIP problem, a model generating program was written in C, and submitted to CPLEX [16] to evaluate a solution. Computational experiments were carried out on a 700MHz Pentium III with 256 Mb of memory.

5.2 Results

In order to illustrate the performance of the LP and MIP models, we consider the following test example to calculate the optimal investment strategy.

An investor has sold a factory and would like to invest in the bank 10 million pounds and to live with this fund for the next eleven years. He would like to get an annual withdrawal of £500,000. *How would he have to invest his wealth in order to maximize the amount he will receive back after eleven years?*

The investment horizon has been considered as 11 years since beyond year 11, the capital gains tax (paid for the capital gains in unit trust) remains unchanged and no more taper relief can be achieved. The length of the investment is a very important fact for the distribution to be chosen. Time periods shorter than 11 years do not make use of the complete taper relief available in unit trust and very longer periods with large withdrawals will make full use of the facilities offered by offshore or onshore bonds about tax deferred withdrawals at the beginning of the investment plan. Besides, the model is highly sensitive to any small change in taxes, costs and bond values, and amount of withdrawals.

We choose the bounds of 43% for the total amount of money invested in every asset independently of the wrapper utilized. This assures the investor a diversification in the portfolio distribution of his investment. Different percentages may be used to reflect investor's different attitudes to risk. In addition, we assume 1.15% annual and 1% transaction costs and no initial cost. Input data used for this example is summarized in Table 1.

TAXES		**Amount** (%)
offshore bond	end of horizon	40
onshore bond	annually	22
onshore bond	end of horizon	18
income tax	cash	40
income tax	equities, bond	25
capital gain	every year	40, 40, 40, 38, 36, 34, 32, 30, 28, 26, 24

Table 1: The input data.

Scenario trees with a specification of four branches at the first time period and only one branching from period 1 to 11 are generated by simulation and optimization based methods. The scenario tree with this topology has 44 nodes, with four different scenarios which is the input to the LP and MIP models. The problem statistics in terms of the number variables, constraints and nonzeros are presented in Table 2.

Since the MIP model allows the investor to use the original capital to pay for his withdrawal, this yields superior return levels. However, even if the difference in the size of the model, as can be seen from Table 2, is not so large, the number of binary variables grows exponentially in the tree for every withdrawal and wrapper considered. Therefore, the MIP problem becomes computationally hard to solve. Even if some examples can be solved in a reasonable amount of CPU time, a small change in input data may sometimes yield a very hard MIP model requiring days of computation. On the other hand, the LP model, even it is a suboptimal of the MIP, can always get an optimal investment strategy in a reasonable amount of CPU time.

	LP Model	**MIP Model**
binary variables	0	120
continuous variables	3250	3730
constraints	1370	1610
nonzeros	9107	10547

Table 2: The problem statistics.

The optimal net redemption values obtained by LP and MIP models presented in Table 3. The mean redemption value is the sum of redemption values multiplied by the branching probabilities at each node of the scenario tree. In addition, the CPU time taken to solve the models is presented in Table 3. Note that the reported optimal redemption values do not reflect the discounted value of the withdrawals taken during the investment period. Therefore, the investor seems to be loosing some money with different risk levels.

	Simulation Based Scenarios			**Optimization Based Scenarios**		
Scen	Prob	LP model	MIP model	Prob	LP model	MIP model
1	0.220188	16,025,148	16,614,691	0.379585	20,921,162	21,002,231
2	0.264875	16,935,782	17,501,384	0.116110	19,184,560	19,242,997
3	0.271813	16,513,642	17,082,077	0.100000	17,493,933	17,595,703
4	0.243125	17,513,249	18,071,496	0.404305	18,545,248	18,621,901
Mean		16,760.942	17,330,798		19,416,208	19,494,934
CPU		15.79	2761.41		16.75	3015.42

Table 3: The optimal net redemption values (pounds) obtained and CPU time (seconds) taken to solve the problem.

It may be also worth noting that banks usually advise the use of only one wrapper. However, the diversification of the investment in different wrappers over the years will assure superior return levels. In order to compare the impacts of two different scenario generation techniques on the post-tax optimization models, we plotted the net portfolio values over the investment horizon for each scenario generated by simulation and optimization techniques. Figures 1 and 2 are the results of LP and MIP models with four return scenarios generated by simulation and optimization based methods, respectively. These figures show that the original investment is distributed among the different wrappers in the four scenarios considered, for both models.

We also consider a comparative illustration of the diversification effect of the scenario tree over assets within each wrapper. The distribution of the original investment on each asset within each wrapper obtained by LP and MIP models with different scenarios are presented in Figures 3 and 4, respectively. Notice that while MIP model ensure the diversification over assets within each wrapper, the LP model with both simulation and optimization return scenarios advice to invest only bonds in the onshore bond wrapper.

A larger scenario tree was also generated to investigate the performance of these models. The event tree with 2 branching at each time period and 11 years planning horizon has 2048 different scenarios and 4094 nodes. The problem statistics in terms of number of variables (including binary), constraints, and nonzero entries for LP and MIP problems are presented in Table 4. For this large model, we were able to solve only the LP model since

	LP Model	**MIP Model**
binary variables	0	6138
continuous variables	243646	268198
constraints	112574	124850
nonzeros	669335	742991

Table 4: The problem statistics.

the MIP problem has 6138 binary variables and could not be solved in several days, even with considerably more computational power. The results presented in Table 5 in terms of net redemption values and CPU time spent to solve the multistage LP problems with scenario trees generated by optimization and simulation based approaches are obtained.

Scenario Generation Methods	**Net Redemption Values**	**CPU time (seconds)**
simulation	15,142,500	228.78
optimization	10,862,700	297.01

Table 5: The results of the LP problem.

Although in Table 5 the optimization based scenario tree took longer to compute an investment strategy, the results in Table 3 indicate that the reverse can also occur. Furthermore, the computational burden does not seem to be a significant factor, in view of the wealth involved. The question, therefore, is whether there is a choice between the scenario tree generators and between the LP and MIP models. The optimization and simulation based trees yield conflicting expected net redemption values in Tables 3 and 5.

Inspecting Figures 1–4, however, indicates that whichever scenario tree is used, the MIP model yields very similar strategies. The wrappers utilised follow very similar trajectories. Hence, although the MIP model is computationally hard, its use seems to dominate any discrepancy between either scenario tree.

6 Conclusions

In this paper, we consider a stochastic programming framework for post-tax portfolio optimization. Two models are considered. The LP model restricts annual withdrawals to be within the amount of investment return in that year. The MIP model allows general withdrawals. Two scenario tree generators are used to specify the dynamic investment problem.

Numerical results indicate that the MIP model is superior in generating better expected net redemption value results and in generating strategies reasonably consistent across scenario trees. Diversification over wrappers with both LP and MIP models is ensured and the original investment is distributed among assets within each wrapper.

Acknowledgements

This research was supported by EPSRC grant GR/M41124. The authors wish to thank John Pottage and Christine Ross from UBS for providing data, and valuable advice.

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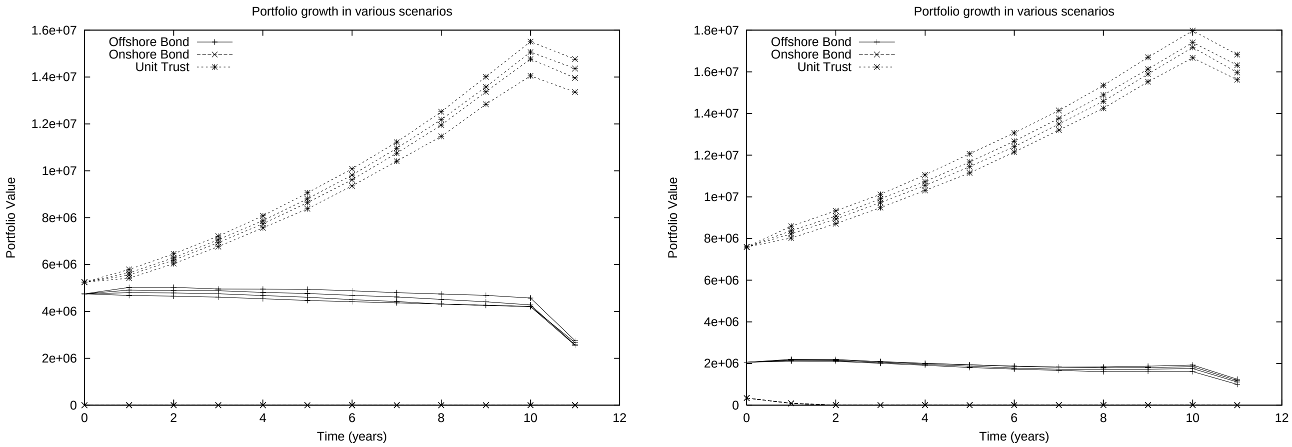

Figure 1: Diversification over wrappers by LP (left) and MIP (right) models with scenarios obtained by simulation

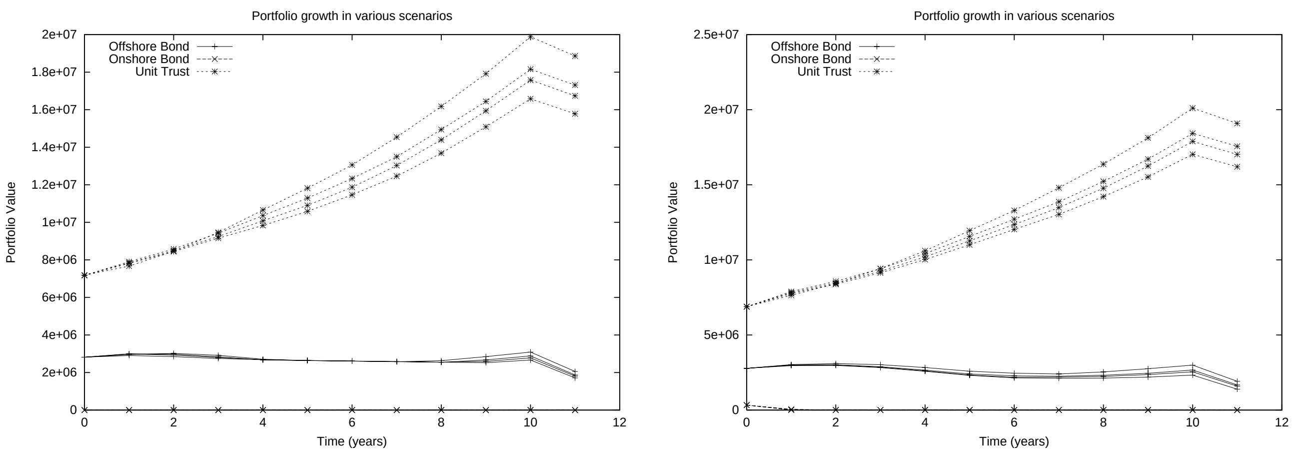

Figure 2: Diversification over wrappers by LP (left) and MIP (right) models with scenarios obtained by optimization

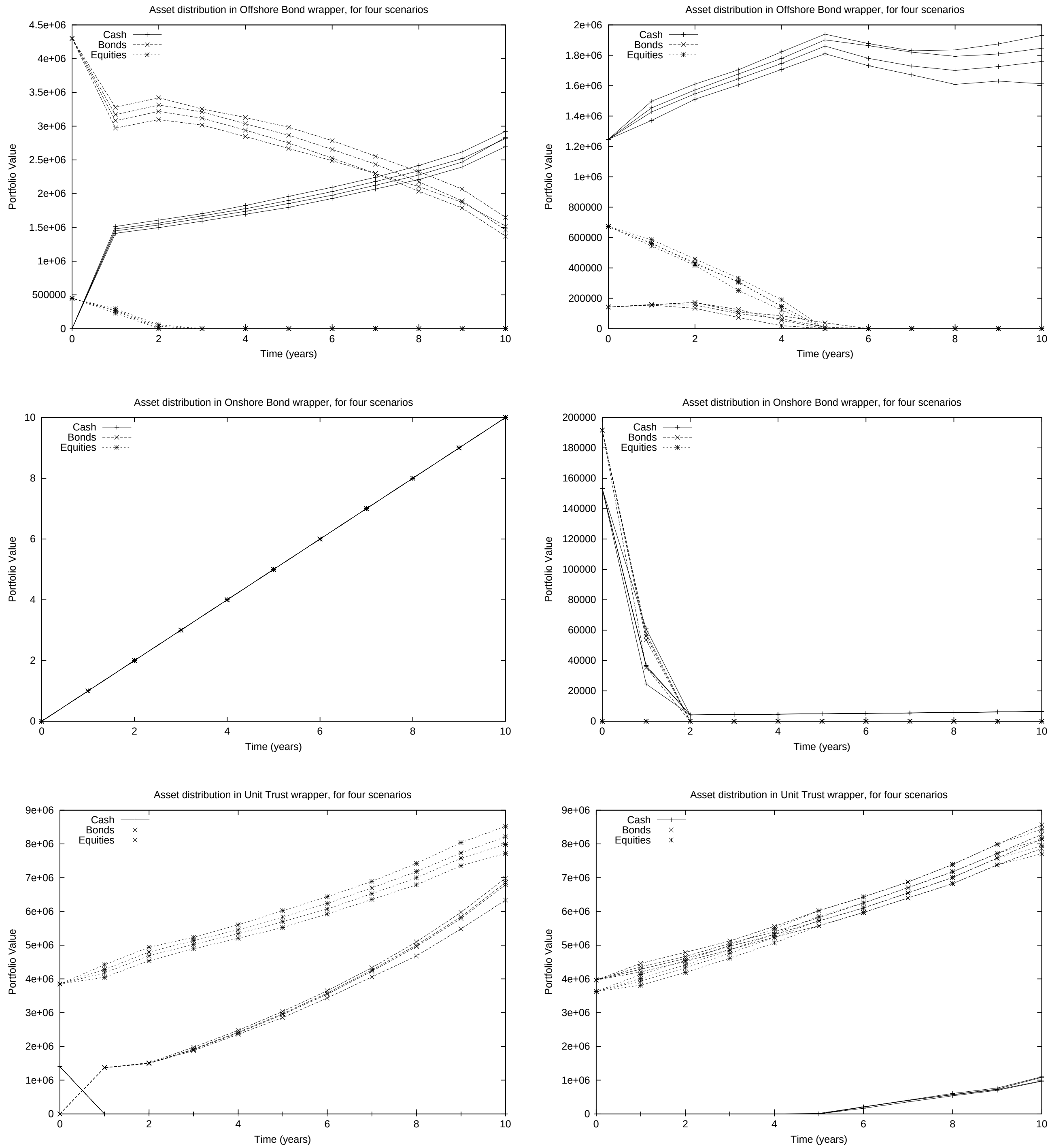

Figure 3: Distribution of assets within each wrapper obtained by LP (left) and MIP (MIP) models with return scenarios generated by simulation based method.

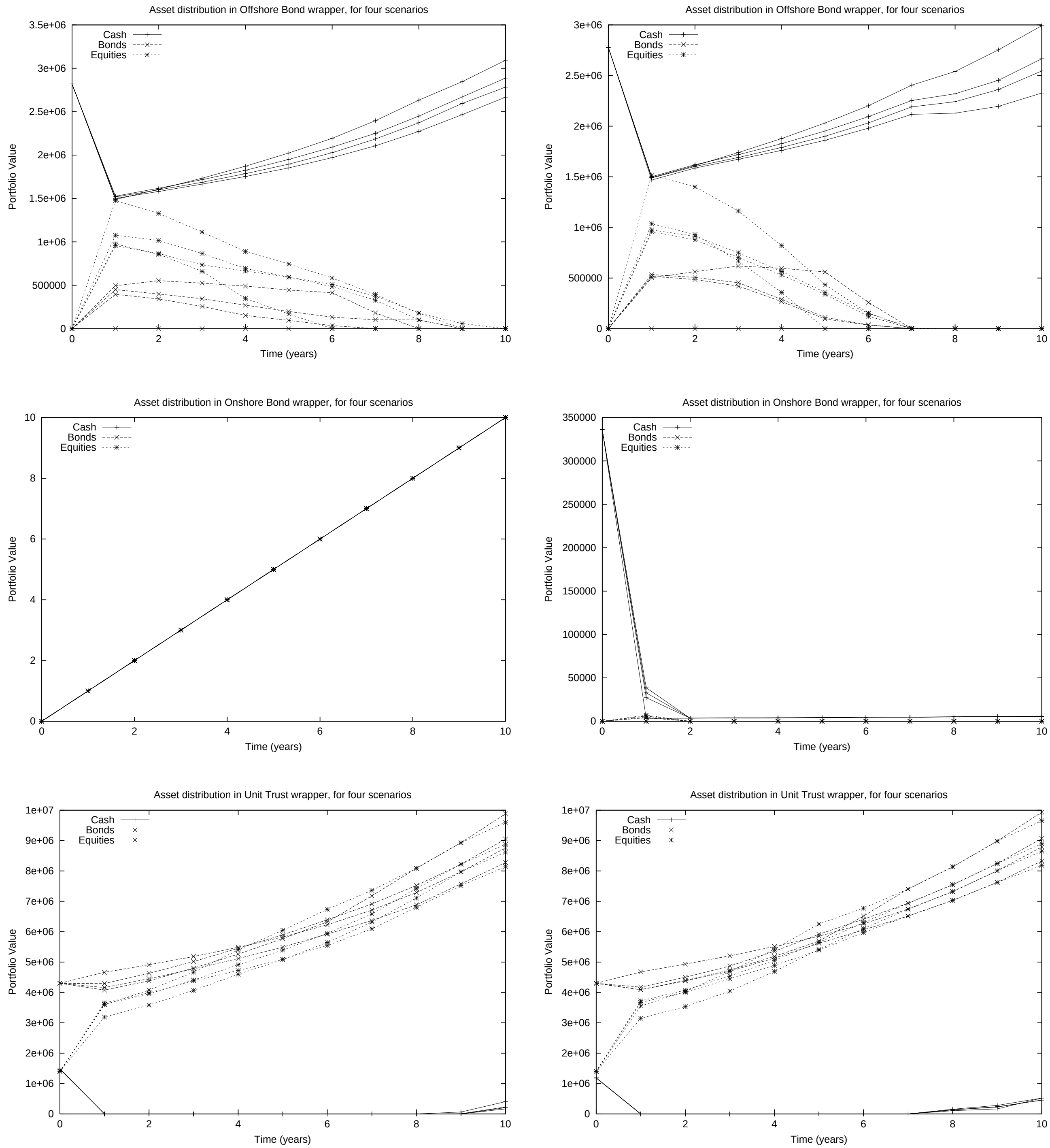

Figure 4: Distribution of assets within each wrapper obtained by LP (left) and MIP (MIP) models with scenarios generated by optimization based method.